ISSN 0735-2727, Radioelectronics and Communications Systems, 2011, Vol. 54, No. 11, pp. 581–591. © Allerton Press, Inc., 2011.
Original Russian Text © A.P. Trifonov, Yu.E. Korchagin, P.A. Kondratovich, 2011, published in Izv. Vyssh. Uchebn. Zaved., Radioelektron., 2011, Vol. 54, No. 11, pp. 3–12.

Efficiency of Estimating Duration of a Signal with Unknown Amplitude

A. P. Trifonov, Yu. E. Korchagin, and P. A. Kondratovich
Voronezh State University, Voronezh, Russia
Received in final form June 17, 2011

Abstract—Quasi-likelihood and maximum likelihood algorithm for estimating duration of a signal with arbitrary share and unknown amplitude are synthesized. Operation efficiency characteristics for the synthesized algorithms are determined. Operation of the synthesized algorithms is verified and applicability limits for asymptotic expressions are obtained using computer emulation

DOI: 10.3103/S073527271111001X

The problem of estimating duration of a signal observed on the noise background is topical in all applications of communications, radar and navigation theory having been discussed in literature many times. Algorithms for estimating duration of a rectangular pulse are studied in [1, 2], while in [3] the same problem is solved for a signal of arbitrary shape. However in a number of practical applications the power of received signal appears to be unknown. Hence is it reasonable to consider algorithms that allow estimating duration of a signal with unknown amplitude. In [4] algorithms for processing the received signal with unknown duration and amplitude are studied only for the case of rectangular pulse. In this paper we solve the problem of synthesizing and analyzing optimal and quasi-optimal algorithms for estimating duration of signals with arbitrary shape and unknown amplitude.

Let's assume that we observe a realization

$$\xi(t) = s(t, a_0, \tau_0) + n(t)$$

of additive mixture containing signal

$$s(t, a_0, \tau_0) = \begin{cases} a_0 f(t), & 0 \le t \le \tau_0, \\ 0, & t < 0, t > \tau_0, \end{cases} \tag{1}$$

and Gaussian white noise $n(t)$ with single-sided spectral density N_0 on the time interval [0, T]. Here a_0 and τ_0 refer to the unknown amplitude and duration of the received signal respectively, while function $f(t)$ describes its shape. Signal's duration takes a value from priory defined interval $\tau \in [T_1, T_2]$. Given the observed realization $\xi(t)$ it is necessary to obtain duration estimate τ_0 for the useful signal (1).

In order to synthesize the duration estimation algorithm we'll use the maximum likelihood (ML) method [1, 2], according to which the duration estimate coincides with position of absolute (greatest) maximum of logarithm of likelihood ratio functional (LRF). However if amplitude and duration are unknown, LRF logarithm depends on two unknown parameters [1, 2]

$$L(a, \tau) = \frac{2a}{N_0} \int_0^\tau \xi(t) f(t) \mathrm{d}t - \frac{a^2}{N_0} \int_0^\tau f^2(t) \mathrm{d}t. \tag{2}$$

Consequently, we have prior parametric ambiguity with respect to amplitude. This ambiguity may be overcome by substituting in expression (2) some of its possible values. These values may be fixed as well as estimated from the observation. In this way one can obtain a series of estimation algorithms (possibly non-optimal). Further we consider estimation algorithms that provide better efficiency but require complex hardware implementation.

One way of overcoming prior parametric ambiguity with respect to amplitude consists in using quasi-likelihood (QL) estimation algorithm. A quasi-likelihood receiver forms LRF logarithm (2) for some expected amplitude a^* and all possible duration values $\tau \in [T_1, T_2]$

$$L^*(\tau) = L(a^*, \tau) = \frac{2a^*}{N_0}\int_0^\tau \xi(t) f(t) \mathrm{d}t - \frac{a^{*2}}{N_0}\int_0^\tau f^2(t)\mathrm{d}t \quad (3)$$

and finds a QL duration estimate as a position of absolute maximum of the decision statistics (3)

$$\tau^* = \arg\sup L^*(\tau). \quad (4)$$

Structure of QL measurer coincides with the structure of maximum likelihood duration estimator for a signal with priory known amplitude [3], except for the amplitude of reference signal which on this case does not equal the true value of the received signal's amplitude.

Let's analyze QL algorithm of duration estimation. According to (3) the random process $L^*(\tau)$ is Gaussian. Hence to describe it we need only expected value and correlation function. By averaging we obtain the expected value

$$S^*(\tau) = \left\langle L^*(\tau) \right\rangle = (1+\delta_\mathrm{a}) q\big(\min(\tau, \tau_0)\big) - (1+\delta_\mathrm{a})^2 q(\tau)/2 \quad (5)$$

and correlation function

$$K(\tau_1, \tau_2) = (1+\delta_\mathrm{a})^2 q\big(\min(\tau_1, \tau_2)\big),$$

where

$$q(\tau) = \frac{2a_0^2}{N_0}\int_0^\tau f^2(t)\mathrm{d}t \quad (6)$$

$q(\tau)$ is signal-to-noise ratio (SNR) at ML receiver's output for a signal with duration τ, while quantity $\delta_\mathrm{a} = (a^* - a_0)/a_0$ denotes relative difference of expected amplitude from its true value. Obviously, when $|\delta_\mathrm{a}| < 1$ the expected value (5) reaches a maximum value in point τ_0. Consequently, in absence of noise the QL estimate (4) coincides with the true value of signal's duration τ_0, and condition $|\delta_\mathrm{a}| < 1$ is a sufficient condition of the QL estimate's consistency.

Let's introduce a new variable $\lambda = q(\tau)$ so that $\lambda \in [\Lambda_1, \Lambda_2]$ and $\Lambda_1 = q(T_1)$, $\Lambda_2 = q(T_2)$. Similarly to [3], we assume that function $f(t)$ describing signal's shape equals zero only on a segment of interval $[0, \tau]$, which has zero measure. Then $q(\tau)$ (6) is a monotonically increasing quantity and the following equality is true: $q(\min(\tau_1, \tau_2)) = \min(q(\tau_1), q(\tau_2))$. Consequently, the decision statistics (3) as a function of variable λ may be represented by

$$L^*(\tau) = L^*[\tau(\lambda)] = \mu^*(\lambda) = (1+\delta_\mathrm{a})\min(\lambda, \lambda_0) - (1+\delta_\mathrm{a})^2 \lambda/2 + \nu(\lambda), \quad (7)$$

where $\lambda_0 = q(\tau_0)$, $\nu(\lambda)$ is Gaussian random process with zero expected value and correlation function

$$B(\lambda_1, \lambda_2) = (1+\delta_\mathrm{a})^2 \min(\lambda_1, \lambda_2), \quad (8)$$

while $\tau(\lambda)$ is determined by solving the equation $q(\tau)=\lambda$.

According to (6) a random quantity

$$\lambda^* = \arg\sup \mu^*(\lambda) \tag{9}$$

is connected with duration estimate (4) by a one-to-one transformation. Consequently, conditional probability density $W_\tau^*(\tau \mid \tau_0)$ of estimate τ^* may be expressed in terms of probability density $W_\lambda^*(\lambda \mid \lambda_0)$ of a random quantity (9). According to [2] the probability density $W_\lambda^*(\lambda|\lambda_0)$ is given by

$$W_\lambda^*\left(\lambda|\lambda_0\right) = \int_{-\infty}^{\infty} \frac{\partial}{\partial\lambda}\left[\left.\frac{\partial F_{21}^*(u,v,\lambda)}{\partial u}\right|_{v=u}\right] \mathrm{d}u, \tag{10}$$

where

$$F_{21}^*(u,v,\Lambda) = P\{\sup_{\Lambda_1 \le \lambda \le \Lambda} \mu^*(\lambda) < u, \sup_{\Lambda < \lambda \le \Lambda_2} \mu^*(\lambda) < v\} \tag{11}$$

$F_{21}^*(u,v,\Lambda)$ is a two-dimensional distribution function of absolute maxima for a random process $\mu^*(\lambda)$. To find the function (11) we'll use the methodology [2]. According to (7), (8) the random process $\mu^*(\lambda)$ is Markovian with displacement and diffusion coefficients

$$k_1 = \frac{1}{2}\begin{cases} 1-\delta_a^2, & \lambda \le \lambda_0, \\ -\left(1+\delta_a\right)^2, & \lambda > \lambda_0, \end{cases}$$

$$k_2 = (1+\delta_a)^2.$$

Consequently, function (11) describes probability that Markovian random process $\mu^*(\lambda)$ will not reach boundaries 0 and u on the interval $\lambda \in [\Lambda_1, \Lambda]$ and boundaries 0 and v on the interval $\lambda \in [\Lambda, \Lambda_2]$. Let's introduce an additional random process

$$y^*(\lambda) = \begin{cases} u - \mu^*(\lambda), & \Lambda_1 \le \lambda \le \Lambda, \\ v - \mu^*(\lambda), & \Lambda < \lambda \le \Lambda_2, \end{cases}$$

which is Gaussian and Markovian with displacement and diffusion coefficients

$$k_1 = \frac{1}{2}\begin{cases} \delta_a^2 - 1, & \lambda \le \lambda_0, \\ \left(1+\delta_a\right)^2, & \lambda > \lambda_0, \end{cases}$$

$$k_2 = (1+\delta_a)^2. \tag{12}$$

Then probability (11)

$$F_{21}^*(u,v,\Lambda) = P\{\sup_{\Lambda_1 \le \lambda \le \Lambda_2} y^*(\lambda) > 0\} \tag{13}$$

represents probability that Markovian random process $y^*(\lambda)$ will not reach boundaries 0 and $+\infty$ on the interval $\lambda \in [\Lambda_1, \Lambda_2]$. According to [5] the searched value (13) may be given by

$$F_{21}^*(u,v,\Lambda) = \int_0^\infty W(y,\Lambda_2)\mathrm{d}y. \tag{14}$$

Here $W(y,\lambda)$ is a solution of Fokker–Planck–Kolmogorov (FPK) equation [2, 5]

$$\frac{\partial W(y,\lambda)}{\partial \lambda} + \frac{\partial}{\partial y}\left[k_1 W(y,\lambda)\right] - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[k_2 W(y,\lambda)\right] = 0 \tag{15}$$

under boundary conditions $W(y=0,\lambda) = W(y=\infty,\lambda) = 0$ and initial condition

$$W(y,\lambda)\big|_{\lambda=\Lambda_1} = \frac{1}{(1+\delta_a)\sqrt{2\pi\Lambda_1}} \exp\left[-\frac{\left(y-u+(1+\delta_a)\Lambda_1-(1+\delta_a)^2\Lambda_1/2\right)^2}{2(1+\delta_a)^2\Lambda_1}\right]. \tag{16}$$

Using mirroring method with sign inversion [5] we obtain solution of equation (15) with coefficients (12) separately for cases $\lambda \in [\Lambda_1, \lambda_0]$ and $\lambda \in (\lambda_0, \Lambda_2]$. Substituting the solutions into expression (14) and then in (10) yields an expression for probability density of random quantity (9)

$$W_\lambda\left(\lambda|\lambda_0\right) = \frac{(1+\delta_a)^2}{2}$$

$$\times\begin{cases} \dfrac{1-\delta_a}{1+\delta_a}\Psi\left[\dfrac{(1-\delta_a^2)(\lambda_0-\lambda)}{2}, \dfrac{(1-\delta_a^2)(\lambda_0-\Lambda_1)}{2}, \dfrac{(1+\delta_a)^2(\Lambda_2-\lambda_0)}{2}, \dfrac{1+\delta_a}{1-\delta_a}\right], & \lambda \le \lambda_0, \\ \Psi\left[\dfrac{(1+\delta_a)^2(\lambda-\lambda_0)}{2}, \dfrac{(1+\delta_a)^2(\Lambda_2-\lambda_0)}{2}, \dfrac{(1-\delta_a^2)(\lambda_0-\Lambda_1)}{2}, \dfrac{1-\delta_a}{1+\delta_a}\right], & \lambda > \lambda_0, \end{cases} \tag{17}$$

where

$$\Psi(y,y_1,y_2,y_3) = \frac{1}{2\sqrt{\pi}y^{3/2}}\left\{\frac{\exp\left[-(y_1-y)/4\right]}{\sqrt{\pi[y_1-y]}} + \Phi\left(\sqrt{\frac{y_1-y}{2}}\right)\right\}$$

$$\times\int_0^\infty x\exp\left\{-\frac{(x+y)^2}{4y}\right\}\left[\Phi\left(\frac{y_2+y_3x}{\sqrt{2y_2}}\right) - \exp(-y_3x)\Phi\left(\frac{y_2-y_3x}{\sqrt{2y_2}}\right)\right]\mathrm{d}x. \tag{18}$$

Since $\lambda = q(\tau)$, from (4) and (9) we obtain $W_\tau(\tau|\tau_0) = W_\lambda(q(\tau)|\,q(\tau_0))q'(\tau)$. Using this expression and considering (18), let's write down expressions for conditional displacement and dispersion for QL duration estimate (4)

$$B\left(\tau^*|\tau_0\right)=\int_{T_1}^{T_2}(\tau-\tau_0)W_\lambda\left(q(\tau)|q(\tau_0)\right)q'(\tau)\mathrm{d}\tau,$$

$$V\left(\tau^*|\tau_0\right)=\int_{T_1}^{T_2}(\tau-\tau_0)^2W_\lambda\left(q(\tau)|q(\tau_0)\right)q'(\tau)\mathrm{d}\tau. \quad (19)$$

Let's consider asymptotic behavior of probability density (17) and displacement and dispersion (19) with increasing SNR. Let's introduce a normalized variable (or a generalized QL duration estimate)

$$\kappa=\frac{(1+\delta_a)^2}{2}\begin{cases}(\lambda-\lambda_0)(1-\delta_a)/(1+\delta_a), & \lambda\le\lambda_0,\\ (\lambda-\lambda_0), & \lambda>\lambda_0,\end{cases} \quad (20)$$

$$\kappa\in[-K_1,K_2],$$

$$K_1=\left[q(\tau_0)-q(T_1)\right](1-\delta_a^2)/2,$$

$$K_2=\left[q(T_2)-q(\tau_0)\right](1+\delta_a)^2/2,$$

with probability density given by

$$W(\kappa)=\begin{cases}\Psi\left[-\kappa,K_1,K_2,(1+\delta_a)/(1-\delta_a)\right], & \kappa\le0,\\ \Psi\left[\kappa,K_2,K_1,(1-\delta_a)/(1+\delta_a)\right], & \kappa>0.\end{cases} \quad (21)$$

Let's assume that in (21) $K_1\to\infty$ and $K_2\to\infty$. This specifically occurs when $\lambda_0=q(\tau_0)\to\infty$ and the values of T_1 and T_2 are fixed. Considering (18) we find a limiting probability density of normalized variable (20)

$$W(\kappa)=\begin{cases}W_0\left(-\kappa,(1+\delta_a)/(1-\delta_a)\right), & \kappa\le0,\\ W_0\left(\kappa,(1-\delta_a)/(1+\delta_a)\right), & \kappa>0,\end{cases} \quad (22)$$

$$W_0(x,y)=\Psi(x,\infty,\infty,y)$$

$$=(2y+1)\exp\{y(y+1)|x|\}\left(1-\Phi\left\{(2y+1)\sqrt{|x|/2}\right\}\right)+\Phi\left\{\sqrt{|x|/2}\right\}-1.$$

It is known from [1, 2] that with increasing SNR λ_0 the QL estimate τ^* converges in mean-square sense to maximum expected value location for the decision statistics (5), which with $|\delta_a|<1$ coincides with the true value of duration τ_0. Hence let's consider the behavior of function $q(\tau)$ in the neighborhood of point τ_0. Let's expand (6) into Taylor series with respect to τ in the neighborhood of point τ_0 limiting our consideration to first infinitesimal order terms

$$q(\tau)\approx q(\tau_0)+2f^2(\tau_0)(\tau-\tau_0)/N_0=q(\tau_0)+\rho_0^2(\tau-\tau_0)/\tau_0,$$

where $\rho_0^2=2f^2(\tau_0)\tau_0/N_0$. Then for normalized variable κ we obtain the following expression:

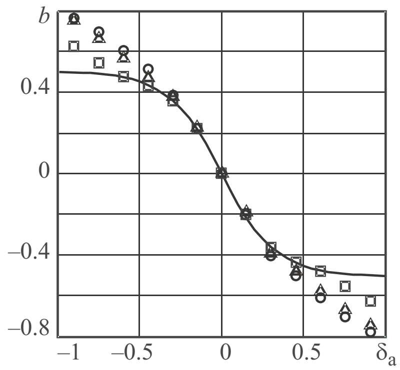

Fig. 1.

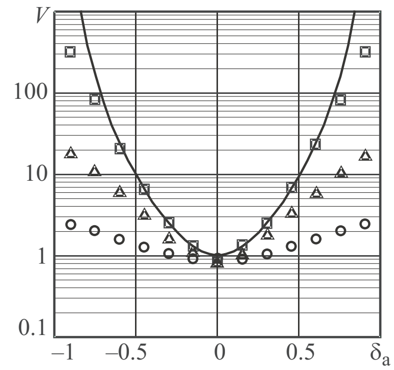

Fig. 2.

$$\kappa = \frac{(1+\delta_a)^2 \rho_0^2}{2\tau_0} \begin{cases} (\tau - \tau_0)(1-\delta_a)/(1+\delta_a), & \tau \le \tau_0, \\ (\tau - \tau_0), & \tau > \tau_0. \end{cases} \tag{23}$$

Using (22) and (23) we may express asymptotic values of duration estimate's displacement and dispersion as follows:

$$B_a\left(\tau^* | \tau_0\right) = \frac{-8\tau_0 \delta_a}{\rho_0^2 (\delta_a - 1)^2 (\delta_a + 1)^2},$$

$$V_a\left(\tau^* | \tau_0\right) = \frac{2\tau_0^2 (13 + 101\delta_a^2 + 15\delta_a^4 - \delta_a^6)}{\rho_0^4 (\delta_a + 1)^4 (\delta_a - 1)^4}. \tag{24}$$

If $a^* = a_0$, then $\delta_a = 0$ and QL duration estimate (4) coincides with ML estimate for a signal with priory known amplitude [3]. When $\delta_a = 0$ expressions (24) appear as follows:

$$B_0 = 0, \quad V_0 = 26\tau_0^2 / \rho_0^4 \tag{25}$$

and coincide with expressions for displacement and dispersion of ML duration estimate for signal with priory known amplitude obtained in [3].

In Figs. 1 and 2 solid lines represent dependences of normalized displacement $b(\delta_a) = B_a(\tau^* | \tau_0) / \sqrt{V_a(\tau^* | \tau_0)}$ and dispersion $v(\delta_a) = V_a(\tau^* | \tau_0) / V_0$ for QL duration estimate on δ_a, respectively. As follows from Figs. 1 and 2, deviation of expected amplitude's value from the true value leads to a significant degradation of estimation quality.

In order to improve precision of duration estimation we can use ML algorithm based on finding position of LRF logarithm's maximum

$$\tau_m = \arg\sup L(\tau), \tag{26}$$

where $L(\tau) = L(a_m, \tau) = \sup\limits_a L(a, \tau)$ is LRF logarithm, where instead of unknown amplitude its maximum likelihood estimate a_m is used, which is identical to maximizing expression (2) with respect to amplitude. Maximization of LRF logarithm (2) with respect to amplitude yields

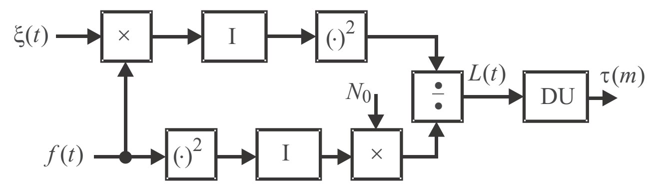

Fig. 3.

$$L(\tau) = \frac{1}{N_0}\left(\int_0^\tau \xi(t) f(t)\mathrm{d}t\right)^2 / \int_0^\tau f^2(t)\mathrm{d}t. \tag{27}$$

Expression (27) defines the structure of receiving unit. Receiver should generate a random process (27) for all possible duration values and calculate ML duration estimate as a position of its maximum. In Fig. 3 a block-diagram of ML duration measurer is depicted with the following denotations: I are integrators on the time interval $[0,t]$, $t \in [0,T_2]$, DU is decision unit searching for position of input signal's maximum on the time interval $[T_1,T_2]$.

Let's consider an auxiliary random process for ML estimation algorithm

$$M(\tau) = \int_0^\tau \xi(t) f(t)\mathrm{d}t. \tag{28}$$

This process is Gaussian and has expected value

$$S_M(\tau) = \langle M(\tau)\rangle = N_0 q(\tau_0)\min\left[q(\tau)/q(\tau_0),1\right]/2a_0$$

and correlation function

$$K_M(\tau_1,\tau_2) = N_0^2 q(\tau_0)\min\left[q(\tau_1)/q(\tau_0), q(\tau_2)/q(\tau_0)\right]/4a_0^2,$$

where function $q(\tau)$ is defined in (6).

In (28) let's pass to a new variable $l = q(\tau)/q(\tau_0)$, $l \in [L_1, L_2]$, $L_1 = q(T_1)/q(\tau_0)$, $L_2 = q(T_2)/q(\tau_0)$. Then for random process (28) as a function of variable l we can write the following expression:

$$M(\tau) = M[\tau(l)] = \chi(l) = \frac{N_0 q(\tau_0)}{2a_0}\left[\min(l,1) + \frac{\omega(l)}{\sqrt{q(\tau_0)}}\right].$$

Here $\tau(l)$ is determined by solving equation $q(\tau)/q(\tau_0) = l$, while $\omega(l)$ is a standard Winner process. Using random process (28), the decision statistics (27) may be represented as a function of variable l

$$L(l) = \frac{2a_0^2\chi^2(l)}{N_0^2 q(\tau_0) l} = z_0^2\frac{\min^2(1,l)}{2l} + z_0\frac{\min(1,l)}{l}\omega(l) + \frac{\omega^2(l)}{2l}, \tag{29}$$

where $z_0^2 = q(\tau_0)$ is SNR for received signal at receiver's output. In case of large SNR the last summand in (29) may be neglected leaving us with approximate expression

$$L(l) \approx z_0^2\frac{\min^2(1,l)}{2l} + z_0\frac{\min(1,l)}{l}\omega(l). \tag{30}$$

Let's perform another substitution in (30): $\lambda = q(\tau_0)l = z_0^2 l$ so that $\lambda \in [\Lambda_1, \Lambda_2]$, $\Lambda_1 = q(T_1)$, $\Lambda_2 = q(T_2)$, $\lambda_0 = q(\tau_0) = z_0^2$. Then the decision statistics (30) as a function of variable λ may be given by

$$L(l) = L[l(\lambda)] = L[\lambda / z_0^2] = \frac{\min^2(\lambda_0,\lambda)}{2\lambda} + \frac{\min(\lambda_0,\lambda)}{\lambda}\omega(\lambda). \qquad (31)$$

This function represents a Gaussian random process with expected value

$$S(\lambda) = \min^2(\lambda_0,\lambda)/2\lambda \qquad (32)$$

and correlation function

$$K(\lambda_1,\lambda_2) = \min(\lambda_1,\lambda_0)\min(\lambda_2,\lambda_0)\min(\lambda_1,\lambda_2)/\lambda_1\lambda_2.$$

The correlation coefficient for decisions statistics (31) $R(\lambda_1,\lambda_2) = \min(\lambda_1,\lambda_2)/\sqrt{\lambda_1\lambda_2}$ satisfied the condition $R(x,y) = R(x,t)R(t,y), x > t > y$ [5, 6]. Consequently, the random process (31) is a Markovian one with displacement and diffusion coefficients [5, 6]

$$k_1(\lambda) = \frac{1}{2}\begin{cases} 1, & \lambda \le \lambda_0, \\ -\lambda_0^2/\lambda^2, & \lambda > \lambda_0, \end{cases}$$

$$k_2(\lambda) = \begin{cases} 1, & \lambda \le \lambda_0, \\ \lambda_0^2/\lambda^2, & \lambda > \lambda_0. \end{cases}$$

Under large SNR maximum of decision statistics (31) is located in the neighborhood of its expected value's maximum [2]. The expected value (32) reaches its maximum value when $\lambda = \lambda_0$. Let's introduce a variable $\varepsilon = (\lambda - \lambda_0)/\lambda_0$, whose absolute value decreases with increasing SNR $\lambda_0 = z_0^2$, and rewrite expression for displacement and diffusion coefficients

$$k_1(\lambda) = \frac{1}{2}\begin{cases} 1, & \lambda \le \lambda_0, \\ -(1+\varepsilon)^{-2}, & \lambda > \lambda_0, \end{cases}$$

$$k_2(\lambda) = \begin{cases} 1, & \lambda \le \lambda_0, \\ (1+\varepsilon)^{-2}, & \lambda > \lambda_0. \end{cases}$$

Since $\varepsilon \to 0$ when $z_0 \to \infty$ the decision statistics (31) in the neighborhood of point $\lambda = \lambda_0$ may be approximated by a Gaussian Markovian process $\mu(\lambda)$ with displacement and diffusion coefficients

$$k_1(\lambda) = \frac{1}{2}\begin{cases} 1, & \lambda \le \lambda_0, \\ -1, & \lambda > \lambda_0, \end{cases}$$

$$k_2(\lambda) = 1. \qquad (33)$$

We'll use this approximation on the whole interval of possible parameter $\lambda \in [\Lambda_1, \Lambda_2]$ values. According to (6) random quantity $\lambda_m = \arg\sup\mu(\lambda)$ is connected with duration estimate τ_m (26) by a one-to-one transformation. Consequently, conditional probability density $W_\tau(\tau|\tau_0)$ of ML duration estimate τ_m may be expressed in terms of probability density $W_\lambda(\lambda|\lambda_0)$ for the random quantity λ_m. An expression similar to

(10) is applicable to probability density $W_\lambda(\lambda|\lambda_0)$, where by $F^*_{21}(u,v,\Lambda)$ we mean two-dimensional distribution function of absolute maxima for random process $\mu(\lambda)$. We should note that when $\delta_a = 0$ displacement and diffusion coefficients (33) for random process $\mu(\lambda)$ coincide with displacement and diffusion coefficients (12) for random process $\mu^*(\lambda)$ (7), while probability density of random quantity $u - \mu(\Lambda_1)$ coincides with probability density (16) of random quantity $u - \mu^*(\Lambda_1)$ when $\delta_a = 0$. Consequently, assuming $\mu(\lambda)$ in (17) we may obtain distribution of maximum's location for random process $\delta_a = 0$

$$W_\lambda\left(\lambda|\lambda_0\right) = \frac{1}{2}\begin{cases}\Psi\left[\dfrac{\lambda_0-\lambda}{2},\dfrac{\lambda_0-\Lambda_1}{2},\dfrac{\Lambda_2-\lambda_0}{2},1\right], & \lambda \le \lambda_0,\\ \Psi\left[\dfrac{\lambda-\lambda_0}{2},\dfrac{\Lambda_2-\lambda_0}{2},\dfrac{\lambda_0-\Lambda_1}{2},1\right], & \lambda > \lambda_0,\end{cases} \tag{34}$$

while the limiting shape of probability density (34) with increasing SNR may be obtained from expression (22) when $\delta_a = 0$

$$W_\lambda\left(\lambda|\lambda_0\right) = W_0\left(\frac{\lambda-\lambda_0}{2}\right)/2,$$

$$W_0(x) = 3\exp\left(2|x|\right)\left\{1-\Phi\left(3\sqrt{|x|/2}\right)\right\} + \Phi\left(\sqrt{|x|/2}\right) - 1.$$

Correspondingly, assuming $\delta_a = 0$ in (24) we obtain asymptotic expressions for displacement and dispersion of ML duration estimate of signal with unknown amplitude

$$B_a\left(\tau_m|\tau_0\right) = 0,$$

$$V_a\left(\tau^*|\tau_0\right) = 26\tau_0^2/\rho_0^4 = 13N_0^2/2a_0^4 f^4(\tau_0).$$

These expressions coincide with similar ones in [3] for ML duration estimate (25) when the signal's amplitude is known. Consequently, absence of prior knowledge on signal's amplitude has no influence on limiting precision of ML estimate of signal's duration. This circumstance allows interpreting dependences in Figs. 1 and 2 as a gain in precision for ML algorithm (26) (Fig. 3) when compared to QL estimation algorithm (4).

In order to verify operation capability of the synthesized estimation algorithms and to determine applicability limits for the obtained asymptotic expressions for estimate's dispersion and displacement statistical simulations of QL and ML algorithms were carried out on a PC for rectangular pulse with inclined vertex [7]. The signal's shape was defined by the following linear function:

$$f(t) = [1 + bt/T_2]/\sqrt{1 + b + b^2/3}.$$

Parameter b characterizes inclination of the vertex. Multiplier $(1 + b + b^2/3)^{-1/2}$ is introduced to retain the same energy for signals with different durations and inclinations. For modeling purposes of the QL algorithm the LRF logarithm (3) was given by

$$L^*(\eta) = S^*(\eta,\eta_0) + N^*(\eta),$$

$$S^*(\eta,\eta_0)=(1+\delta_a)z_r^2\ \min(\eta,\eta_0)\frac{1+b\min(\eta,\eta_0)+b^2\min^2(\eta,\eta_0)/3}{1+b+b^2/3}$$

$$-(1+\delta_a)^2 z_r^2\eta\frac{1+b\eta+b^2\eta^2/3}{2(1+b+b^2/3)},$$

$$N^*(\eta)=z_r(1+\delta_a)\sqrt{\frac{T_2}{N_0}}\int_0^\eta n(T_2x)(1+bx)dx\Big/\sqrt{1+b+b^2/3},$$

where $\eta=\tau/T_2$, $z_r^2=2a_0^2T_2/N_0$ is maximum possible SNR for the received signal. During modeling with step $\Delta\eta=10^{-6}$ samples of function $N^*(\eta)$ were generated, using which implementation of LRF logarithm was approximated using step function with maximum mean-square error of $\varepsilon=0.1$. Discrete samples of LRF logarithm were represented by

$$L^*(n\Delta\eta)=S^*(n\Delta\eta,n_0\Delta\eta)+z_r(1+\delta_a)\sqrt{\Delta\eta/2}$$

$$\times\sum_{k=1}^{n}(1+bk\Delta\eta)x_k\Big/\sqrt{1+b+b^2/3},$$

where x_k are Gaussian statistically independent quantities with zero expected value and unit dispersion, $n=\overline{n_1,n_2}$, $n_1=1/\gamma\Delta\eta$, $n_2=1/\Delta\eta$, $n_0=\eta_0/\Delta\eta$, $\gamma=T_2/T_1$ is dynamic range of unknown duration change. During ith experiment using samples $L_i^*(n\Delta\eta)$ the following quantities were generated

$$n_i^*=\arg\sup L_i^*(n\Delta\eta),\quad \eta_i^*=n_i^*\Delta\eta.$$

In the process of modeling $N=10^5$ tests were conducted. Experimental values for QL duration estimate's displacement and dispersion were calculated using the following formulas:

$$B^*=\frac{T_2\Delta\eta}{N}\sum_{i=1}^{N}(n_i^*-n_0),$$

$$V^*=\frac{T_2^2\Delta\eta^2}{N}\sum_{i=1}^{N}(n_i^*-n_0)^2.$$

For modeling purposes of the ML algorithm the LRF logarithm (27) was given by

$$L(\eta)=\frac{\left[z_rS(\eta,\eta_0)+N(\eta)\right]^2}{2(1+b\eta+b^2\eta^2/3)},$$

$$S(\eta,\eta_0)=\min(\eta,\eta_0)$$

$$\times\left[1+b\min(\eta,\eta_0)+b^2\min^2(\eta,\eta_0)/3\right]\Big/\sqrt{1+b+b^2/3},$$

$$N(\eta)=\sqrt{\frac{T_2}{N_0}}\int_0^{\eta} n(T_2 x)(1+bx)\mathrm{d}x.$$

During modeling with step $\Delta\eta = 10^{-6}$ samples of function $N(\eta)$ were generated, using which implementation of LRF logarithm was approximated using step function with maximum mean-square error of $\varepsilon = 0.1$. Discrete samples of LRF logarithm were represented by

$$L(n\Delta\eta)=\frac{\left[z_r S(n\Delta\eta, n_0\Delta\eta)+\sqrt{\Delta\eta/2}\sum_{k=1}^{n}(1+bk\Delta\eta)x_k\right]^2}{2(1+bn\Delta\eta+b^2n^2\Delta\eta^2/3)}.$$

During ith experiment using these samples the following quantities were generated

$$n_{mi}=\mathrm{argsup} L_i(n\Delta\eta),\quad \eta_{mi}=n_{mi}\Delta\eta.$$

In the process of modeling $N=10^5$ tests were conducted. Experimental values for ML duration estimate's displacement and dispersion were calculated using the following formulas:

$$B_m=\frac{T_2\Delta\eta}{N}\sum_{i=1}^{N}(n_{mi}-n_0),$$

$$V_m=\frac{T_2^2\Delta\eta^2}{N}\sum_{i=1}^{N}(n_{mi}-n_0)^2.$$

Statistical simulation modeling results are depicted in Figs. 1 and 2. Circles, squares and triangles denote dependences of normalized displacement $B^*/\sqrt{V^*}$ and dispersion V^*/V_m, obtained during modeling for rectangular pulse with inclined vertex $b=1$ in case of various SNR values. Circles, squares and triangles correspond to SNR $z_r=5$, $z_r=10$, and $z_r=20$, respectively. It was assumed that true value of signal's duration lies within the priory known interval, while dynamic range of duration change was $\gamma=8$. As follows from Figs. 1 and 2, with increasing SNR experimental dependences approach the theoretical ones, and the range of possible δ_a values providing adequate description of experimental values by asymptotic expression for displacement and dispersion increase as well.

The obtained expressions for characteristics of different duration estimation algorithms for a signal with unknown amplitude allow making a well-grounded decision when choosing the estimation algorithm to be used depending in the available information on signal's amplitude and requirements to simplicity of implementation and its precision.

REFERENCES

1. V. I. Tikhonov, *Optimal Signal Reception* (Radio i Svyaz', Moscow, 1983) [in Russian].
2. A. P. Trifonov and Yu. S. Shinakov, *Mutual Signals Discrimination and Estimation of Their Parameters on the Interference Background* (Radio i Svyaz', Moscow, 1986) [in Russian].
3. A. P. Trifonov and Yu. E. Korchagin, "Reception of signal with unknown duration," Izv. Vyssh. Uchebn. Zaved., Radiofizika **45**, No. 7, 625 (2002).
4. A. P. Trifonov and V. K. Buteiko, "Reception of signal with unknown amplitude and duration on the white noise background," Izv. Vyssh. Uchebn. Zaved., Radioelektron. **27**, No. 8, 28 (1981); Radioelectron. Commun. Syst. **27**, No. 8, 23 (1981).
5. V. I. Tikhonov and M. A. Mironov, *Markovian Processes* (Radio i Svyaz', Moscow, 1977) [in Russian].
6. Yu. G. Sosulin, *Detection and Estimation Theory of Stochastic Signals* (Sov. Radio, Moscow, 1978) [in Russian].
7. M. I. Gryaznov, M. L. Gurevich, and Yu. A. Ryabinin, *Pulse Parameters Measurements* (Radio i Svyaz', Moscow, 1991) [in Russian].