Radiophysics and Quantum Electronics, Vol. 55, No. 6, November, 2012
(Russian Original Vol. 55, No. 6, June, 2012)

ESTIMATION OF THE APPEARANCE AND DISAPPEARANCE TIMES OF UNKNOWN-AMPLITUDE SIGNALS

A. P. Trifonov, * **Yu. É. Korchagin, and P. A. Kondratovich**

UDC 621.321

We synthesize quasi-likelihood, maximum-likelihood, and quasi-optimal algorithms for estimating the appearance and disappearance times of an arbitrary-shaped unknown-amplitude signal. The asymptotic characteristics of the estimates are found. The synthesized algorithms are statistically simulated on a computer.

1. INTRODUCTION

The problem of estimating the times of appearance and disappearance of a signal observed against the noise background is topical for many applications of statistical radiophysics, radar, and seismology [1–5]. The problem of estimating the times of a jump-like signal variation is considered in [2]. However, the algorithms of [2] require solution of complicated nonlinear stochastic differential equations and use a large amount of *a priori* information. The algorithms for estimating the appearance and disappearance times of a rectangular pulse are studied in [3], whereas an arbitrary-shaped deterministic signal with unknown (or arbitrary) appearance and disappearance times, which is observed against the background of additive Gaussian white noise in the case of continuous observation time is discussed in [4, 5]. However, the received-signal power is often unknown in practical applications. Therefore, it is expedient to consider algorithms for estimating the appearance and disappearance times of a signal with unknown amplitude. In this work, using the maximum-likelihood method, we synthesize algorithms for estimating the appearance and disappearance times of an arbitrary-shaped deterministic signal with unknown amplitude. For the synthesized algorithms, we find their operation-efficiency characteristics whose accuracy increases with increasing signal-to-noise ratio (SNR).

The signal with unknown appearance and disappearance times can be written as

$$s(t,a,\theta_1,\theta_2)=\begin{cases} af(t), & \theta_1\le t\le\theta_2;\\ 0, & t<\theta_1, t>\theta_2,\end{cases} \tag{1}$$

where $f(t)$ is an *a priori* known continuous function which describes the signal shape, a is the amplitude, and θ_1 and θ_2 are the unknown appearance and disappearance times, respectively, which take their values in the *a priori* intervals

$$\theta_i\in[\theta_{i\,\min},\theta_{i\,\max}],\qquad i=1,2.$$

To exclude that the signal can disappear before its appearance, we put $\theta_{1\,\max}<\theta_{2\,\min}$. It is assumed that the function describing the signal shape satisfies the condition $f(\theta_i)\ne 0$.

* trif@rf.phys.vsu.ru

Voronezh State University, Voronezh, Russia. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 55, No. 6, pp. 455–471, June 2012. Original article submitted April 27, 2012; accepted June 26, 2012.

An arbitrary process

$$\xi(t) = s(t, a_0, \theta_{01}, \theta_{02}) + n(t)$$

, which is observed in the time interval $[0, T]$, is an additive mixture of the useful signal $s(a_0, \theta_{01}, \theta_{02})$ and Gaussian white noise $n(t)$ with one-sided spectral density N_0. Here, a_0, θ_{01} and θ_{02} are the true values of the amplitude and the appearance and disappearance times, respectively, which are unknown at the reception point. On the basis of the observed process $\xi(t)$, the receiver should form the estimates of the appearance and disappearance times of useful signal (1).

If the amplitude of useful signal (1) is *a priori* known, one can use an estimation algorithm based on the maximum-likelihood method, which was synthesized in [4]. According to this algorithm, the estimates of the appearance and disappearance times coincide with the coordinates of the maximum of an logarithm of the likelihood-ratio functional (LRF) [6]:

$$L(\theta_1, \theta_2) = \frac{2a_0}{N_0} \int_{\theta_1}^{\theta_2} \xi(t) f(t)\, \mathrm{d}t - \frac{a_0^2}{N_0} \int_{\theta_1}^{\theta_2} f^2(t)\, \mathrm{d}t.$$

Hereafter, the first term is the stochastic integral in the Ito sense. The characteristics of the joint maximum-likelihood estimates of the appearance and disappearance times, i.e., the probability densities, biases, and variances were obtained in [5]. However, if both the appearance and disappearance times and the amplitude are unknown, then the LRF logarithm depends on three unknown parameters [6]

$$L(a, \theta_1, \theta_2) = \frac{2a}{N_0} \int_{\theta_1}^{\theta_2} \xi(t) f(t)\, \mathrm{d}t - \frac{a^2}{N_0} \int_{\theta_1}^{\theta_2} f^2(t)\, \mathrm{d}t. \tag{2}$$

If the unknown amplitude a in Eq. (2) is replaced by its certain values, then one can obtain some (probably, nonoptimal) algorithms for estimating the appearance and disappearance times. These values of the amplitudes fixed or can be determined from realizations of the observed data. The resulting estimation algorithms, which are considered below, differ in their efficiency and degree of simplicity of the hardware or software realization.

2. QUASI-LIKELIHOOD ESTIMATION ALGORITHM

Using the quasi-likelihood estimation algorithm is a way for overcoming the *a priori* parametric uncertainty with respect to the amplitude [7]. The quasi-likelihood receiver generates the LRF logarithm given by Eq. (2) for some expected amplitude a^* and all possible appearance and disappearance times

$$L^*(\theta_1, \theta_2) = \frac{2a^*}{N_0} \int_{\theta_1}^{\theta_2} \xi(t) f(t)\, \mathrm{d}t - \frac{a^{*2}}{N_0} \int_{\theta_1}^{\theta_2} f^2(t)\, \mathrm{d}t. \tag{3}$$

Then the receiver finds the quasi-likelihood estimates of the appearance and disappearance times as points of the absolute (largest) maximum of the decision statistic in Eq. (3):

$$(\theta_1^*, \theta_2^*) = \arg\sup L^*(\theta_1, \theta_2). \tag{4}$$

According to Eq. (4), the receiver should generate a two-dimensional random field given by Eq. (3) for all possible values of the unknown appearance and disappearance times. Therefore, its hardware realization turns out to be sufficiently complicated in the general case. Indeed, search for the quantities in Eq. (4) assumes the development of a structure that is multichannel with respect to both unknown parameters. However, the hardware-realization difficulties for the quasi-likelihood estimation algorithm given by Eq. (4)

can partially be avoided if, by analogy with [4], the random field in Eq. (3) is represented as the sum $L^*(\theta_1, \theta_2) = L_1^*(\theta_1) + L_2^*(\theta_2)$ of two random processes. The first process depends only on the appearance time θ_1, while the second process depends only on the disappearance time θ_2

$$L_1^*(\theta_1) = \frac{2a^*}{N_0} \int\limits_{\theta_1}^{\theta} \xi(t) f(t)\,\mathrm{d}t - \frac{a^{*2}}{N_0} \int\limits_{\theta_1}^{\theta} f^2(t)\,\mathrm{d}t, \tag{5}$$

$$L_2^*(\theta_2) = \frac{2a^*}{N_0} \int\limits_{\theta}^{\theta_2} \xi(t) f(t)\,\mathrm{d}t - \frac{a^{*2}}{N_0} \int\limits_{\theta}^{\theta_2} f^2(t)\,\mathrm{d}t, \tag{6}$$

where θ is an arbitrary point which belongs to the interval $(\theta_{1\max}, \theta_{2\min})$.

According to Eqs. (5) and (6), the random processes $L_1^*(\theta_1)$ and $L_2^*(\theta_2)$ are statistically independent since they are integrals of white noise over the nonoverlapping intervals. Therefore, the locations of maxima of the random field $L^*(\theta_1, \theta_2)$ with respect to the variables θ_1 and θ_2 coincide with the locations of maxima of the random processes $L_1^*(\theta_1)$ and $L_2^*(\theta_2)$, respectively. As a result, for the quasi-likelihood estimates of the appearance and disappearance times, we can write

$$\theta_j^* = \arg\sup L_j^*(\theta_j), \qquad \theta_j \in [\theta_{j\min}, \theta_{j\max}],$$

where $j = 1, 2$. The block diagram of a quasi-likelihood meter of the appearance and disappearance times coincides with that of a maximum-likelihood meter described in [4] (see the dashed part of Fig. 1 in [4]), where the product $a^* f(t)$ should be used instead of the function $f(t)$. Writing the decision statistic as a sum of two statistically independent random processes, we can not only propose a sufficiently simple hardware realization of the quasi-likelihood meter, but also analyze the quasi-likelihood algorithm for estimation according to the method of [5]. For a complete statistical description of the decision statistic, it suffices to find the mathematical expectations and the correlation functions of Gaussian independent random processes (5) and (6). Performing the averaging, we obtain the mathematical expectations

$$S_1^*(\theta_1) = \langle L_1^*(\theta_1)\rangle = (1+\delta_a)\, Q[\max(\theta_{01}, \theta_1), \theta] - (1+\delta_a)^2\, Q(\theta_1, \theta)/2,$$
$$S_2^*(\theta_2) = \langle L_2^*(\theta_2)\rangle = (1+\delta_a)\, Q[\theta, \min(\theta_{02}, \theta_2)] - (1+\delta_a)^2\, Q(\theta, \theta_2)/2$$

and the correlation functions

$$B_1^*(\theta_{11}, \theta_{21}) = \langle [L_1^*(\theta_{11}) - S_1^*(\theta_{11})]\,[L_1^*(\theta_{21}) - S_1^*(\theta_{21})]\rangle = (1+\delta_a)^2\, Q[\max(\theta_{11}, \theta_{21}), \theta],$$
$$B_2^*(\theta_{12}, \theta_{22}) = \langle [L_2^*(\theta_{12}) - S_2^*(\theta_{12})][L_2^*(\theta_{22}) - S_2^*(\theta_{22})]\rangle = (1+\delta_a)^2\, Q[\theta, \min(\theta_{12}, \theta_{22})],$$

where $\delta_a = (a^* - a_0)/a_0$ is a quantity characterizing the relative deviation of the expected amplitude a^* from its true value a_0. In what follows, the quantity δ_a will be called the detuning of the quasi-likelihood meter with respect to the amplitude, and the quantity

$$Q(\theta_1, \theta_2) = \frac{2a_0^2}{N_0} \int\limits_{\theta_1}^{\theta_2} f^2(t)\,\mathrm{d}t \tag{7}$$

will be called the SNR at the output of the maximum-likelihood receiver for the received signal with the appearance and disappearance times θ_1 and θ_2, respectively.

Let $f(t)$ can turn to zero only in a zero-measure part of the interval $[\theta_{1\min}, \theta_{2\max}]$. Then $Q(\theta_1, \theta)$ is a monotonically decreasing function of the argument θ_1, $Q(\theta, \theta_2)$ is a monotonically increasing function of θ_2 and the equalities $Q(x, \theta) = -Q(\theta, x)$, $Q[\max(x, y), \theta] = \min[Q(x, \theta), Q(y, \theta)]$, and $Q[\theta, \min(x, y)] =$

$\min[Q(\theta,x),Q(\theta,y)]$ take place. Using the properties of function (7), we can rewrite the mathematical expectations and the correlation functions of the processes given by Eqs. (5) and (6) in the form

$$S_j^*(\theta_j) = (1+\delta_a)\min[(-1)^j\,Q(\theta,\theta_{0j}),(-1)^j\,Q(\theta,\theta_j)] - (1+\delta_a)^2\,(-1)^j\,Q(\theta,\theta_j)/2,$$
$$B_j^*(\theta_{1j},\theta_{2j}) = (1+\delta_a)^2\min[(-1)^j\,Q(\theta,\theta_{1j}),(-1)^j\,Q(\theta,\theta_{2j})]. \tag{8}$$

Hereafter, we assume $j=1,2$. It is easily seen that for $|\delta_a|<1$, the mathematical expectations given by Eq. (8) attain their maxima at points which coincide with the true values θ_{0j} of the unknown appearance and disappearance times.

Let us pass in Eqs. (5) and (6) from the variables θ_1 and θ_2 to new variables $\lambda_j = (-1)^j\,Q(\theta,\theta_j)$ such that $\lambda_j\in[\lambda_{j\min},\lambda_{j\max}]$, $\lambda_{1\min}=Q(\theta_{1\max},\theta)$, $\lambda_{1\max}=Q(\theta_{1\min},\theta)$, $\lambda_{2\min}=Q(\theta,\theta_{2\min})$, and $\lambda_{1\max}=Q(\theta,\theta_{2\max})$. Then for the random processes given by Eqs. (5) and (6), we can write

$$L_j^*(\theta_j) = L_j^*[g_j(\lambda_j)] = \mu_j^*(\lambda_j) = (1+\delta_a)\min(\lambda_j,\lambda_{0j}) - (1+\delta_a)^2\,\lambda_j/2 + \nu_j(\lambda_j), \tag{9}$$

where $\lambda_{01}=Q(\theta_{01},\theta)$, $\lambda_{02}=Q(\theta,\theta_{02})$, and $\nu_j(\lambda_j)$ are the statistically independent Gaussian random processes with zero mathematical expectations and the correlation functions

$$B_j^*(\lambda_{1j},\lambda_{2j}) = (1+\delta_a)^2\min(\lambda_{1j},\lambda_{2j}), \tag{10}$$

while $g_j(\lambda_j)$ are solutions of the equations $(-1)^j\,Q(\theta,\theta_j)=\lambda_j$ for θ_j. According to the properties of the function in Eq. (7), the locations of the maxima of the random processes given by Eq. (9) are written as

$$\lambda_j^* = \arg\sup\mu_j(\lambda_j), \tag{11}$$

and are related to the estimates of the appearance and disappearance times by one-to-one transformations. Therefore, the conditional probability densities $W_{\theta j}^*(\theta_j\mid\theta_{0j})$ of the quasi-likelihood estimates of the appearance and disappearance times can be expressed in terms of the probability densities $W_{\lambda j}^*(\lambda_j\mid\lambda_{0j})$ of the random quantities given by Eq. (11):

$$W_{\theta j}^*(\theta_j\mid\theta_{0j}) = W_{\lambda j}^*\left[(-1)^j\,Q(\theta,\theta_j)\,\middle|\,(-1)^j\,Q(\theta,\theta_{0j})\right]\left|\frac{\partial Q(\theta,\theta_j)}{\partial\theta_j}\right|. \tag{12}$$

By analogy with [5, 8], the probability densities $W_{\lambda j}^*(\lambda_j\mid\lambda_{0j})$ are written as

$$W_{\lambda j}^*(\lambda_j\mid\lambda_{0j}) = \int\limits_{-\infty}^{+\infty}\frac{\partial}{\partial\lambda_j}\left[\left.\frac{\partial F_{2j}(u,v,\lambda_j)}{\partial u}\right|_{u=v}\right]\mathrm{d}u, \tag{13}$$

where

$$F_{2j}(u,v,x) = P\left[\sup_{\lambda_{j\min}\le\lambda_j<x}\mu_j^*(\lambda_j)<u,\ \sup_{x\le\lambda_j\le\lambda_{j\max}}\mu_j^*(\lambda_j)<v\right] \tag{14}$$

are the two-dimensional distribution functions of values of the absolute maxima of the random processes $\mu_j^*(\lambda_j)$. According to Eqs. (9) and (10), $\mu_j^*(\lambda_j)$ are the statistically independent Gaussian Markov random processes [9] with the drift and diffusion coefficients

$$k_{1j}(\gamma_0) = \frac{1}{2}\times\begin{cases}1-\delta_a^2, & \lambda_{j\min}\le\lambda_j\le\lambda_{0j};\\ -(1+\delta_a)^2, & \lambda_{0j}<\lambda_j\le\lambda_{j\max},\end{cases}\qquad k_{2j}=(1+\delta_a)^2. \tag{15}$$

Therefore, by analogy with [8], for the function of Eq. (14) we can write

$$F_{2j}(u,v,x) = \int\limits_{-\infty}^{v} W_j(y,\lambda_{j\max})\,\mathrm{d}y, \tag{16}$$

where $W_j(y, \lambda_j)$ are the solutions of the Fokker–Planck–Kolmogorov (FPK) equation [9]

$$\frac{\partial W_j(y,\lambda_j)}{\partial \lambda_j} + \frac{\partial}{\partial y}[k_{1j} W_j(y,\lambda_j)] - \frac{1}{2}\frac{\partial^2}{\partial y^2}[k_{2j} W_j(y,\lambda_j)] = 0 \tag{17}$$

with the drift and diffusion coefficients, which are given by Eq. (15), for the initial conditions

$$W_j(y, \lambda_j = \lambda_{j\,\min}) = \exp\left\{-\left[y + \lambda_{j\,\min}\delta_a\,(1+\delta_a)\right]^2 / \left[2\,(1+\delta_a)^2\,\lambda_{j\,\min}\right]\right\} \Big/ \left[(1+\delta_a)\,\sqrt{2\pi\lambda_{j\,\min}}\right]$$

and the boundary conditions $W_j(y = u, \lambda_j) = W_j(y = -\infty, \lambda_j) = 0$ for $\lambda_j \in [\lambda_{j\,\min}, x]$ and $W_j(y = v, \lambda_j) = W_j(y = -\infty, \lambda_j) = 0$ for $\lambda_j \in [x, \lambda_{j\,\max}]$.

Using the reflection method with the sign reversal [9], we find the solution of Eq. (17) with coefficients (15) individually for the cases $\lambda_j \in [\lambda_{j\,\min}, \lambda_{0j}]$ and $\lambda_j \in [\lambda_{0j}, \lambda_{j\,\max}]$. Substituting the obtained solutions into Eq. (16) and then Eq. (16) into Eq. (13), by analogy with [5], we obtain the expressions for the probability density of the random quantity given by Eq. (11):

$$W^*_{\lambda j}(\lambda_j \mid \lambda_{0j}) = \begin{cases} (1-\delta_a^2)\,\Psi\big[(1-\delta_a^2)\,(\lambda_{0j}-\lambda_j), (1-\delta_a^2)\,(\lambda_{0j}-\lambda_{j\,\min}), \\ (1+\delta_a)^2\,(\lambda_{j\,\max}-\lambda_{0j}), (1+\delta_a)/(1-\delta_a)\big]\,, \qquad \lambda_j \le \lambda_{0j}; \\ (1+\delta_a)^2\,\Psi\big[(1+\delta_a)^2\,(\lambda_j-\lambda_{0j}), (1+\delta_a)^2\,(\lambda_{j\,\max}-\lambda_{0j}), \\ (1-\delta_a^2)\,(\lambda_{0j}-\lambda_{j\,\min}), \qquad (1-\delta_a)/(1+\delta_a)\big]\,, \qquad \lambda_j > \lambda_{0j}, \end{cases} \tag{18}$$

where

$$\Psi(y, y_1, y_2, y_3) = \left\{\Phi\left(\sqrt{\frac{y_1-y}{4}}\right) + \exp\left[-\frac{y_1-y}{8}\right] \Big/ \sqrt{\frac{\pi\,(y_1-y)}{2}}\right\} \frac{1}{|y|^{3/2}\sqrt{2\pi}}$$
$$\times \int_0^\infty x \exp\left[-\frac{(x+y/2)^2}{2y}\right] \left[\Phi\left(\frac{y_3 x + y_2/2}{\sqrt{y_2}}\right) - \exp(-y_3 x)\Phi\left(\frac{-y_3 x + y_2/2}{\sqrt{y_2}}\right)\right] \mathrm{d}x. \tag{19}$$

In this case, the probability density in Eq. (18) is independent of choosing the value of θ in Eqs. (5) and (6). On the basis of Eqs. (12) and (18), we write the expressions for the conditional biases and variances of the estimates of the appearance and disappearance times in Eq. (4):

$$B(\theta^*_j \mid \theta_{0j}) = \int_{\theta_{j\,\min}}^{\theta_{j\,\max}} (\theta_j - \theta_{0j})\, W^*_{\lambda j}\left[(-1)^j\, Q(\theta, \theta_j) \,\Big|\, (-1)^j Q(\theta, \theta_{0j})\right] \left|\frac{\partial Q(\theta,\theta_j)}{\partial \theta_j}\right| \mathrm{d}\theta_j, \tag{20}$$

$$V(\theta^*_j \mid \theta_{0j}) = \int_{\theta_{j\,\min}}^{\theta_{j\,\max}} (\theta_j - \theta_{0j})^2\, W^*_{\lambda j}\left[(-1)^j\, Q(\theta, \theta_j) \,\Big|\, (-1)^j\, Q(\theta, \theta_{0j})\right] \left|\frac{\partial Q(\theta,\theta_j)}{\partial \theta_j}\right| \mathrm{d}\theta_j. \tag{21}$$

Assuming $\delta_a = 0$ and $y_3 = 1$ in Eqs. (18) and (19), respectively, we see that Eqs. (18), (20), and (21) are transformed to exact formulas for the characteristics of the maximum-likelihood estimate of the appearance and disappearance times for a signal with the *a priori* known amplitude, which were obtained in [5].

Asymptotic behavior of the probability densities (18), biases (20) and variances (21) with increasing SNR is studied in [5] for $\delta_a = 0$. Performing similar transformations for the normalized variables (or generalized quasi-likelihood estimates of the appearance and disappearance times)

$$\kappa_j = \begin{cases} (1-\delta_a^2)\,(\lambda_j - \lambda_{0j}), & \lambda_j \le \lambda_{0j}; \\ (1+\delta_a)^2\,(\lambda_j - \lambda_{0j}) & \lambda_j > \lambda_{0j}, \end{cases} \tag{22}$$

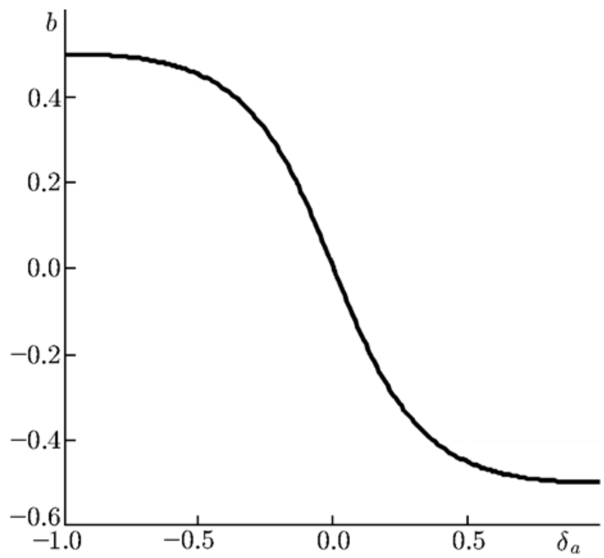

Fig. 1. Normalized bias of the estimates of the appearance and disappearance times.

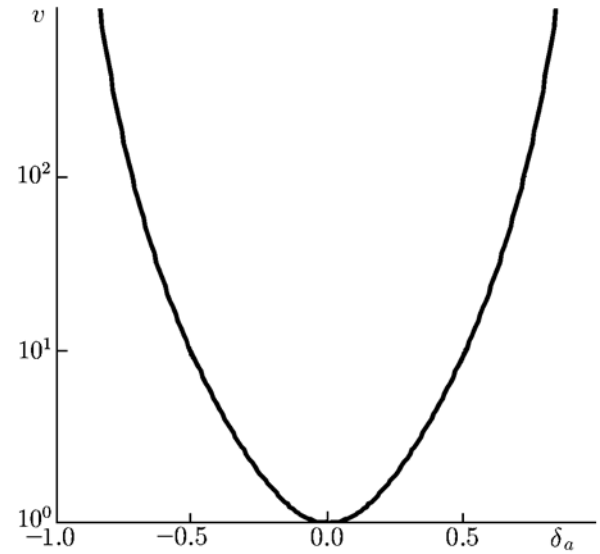

Fig. 2. Normalized variance of the estimates of the appearance and disappearance times.

we obtain the limiting probability density with increasing SNR in the form

$$W_j(\kappa_j) = \begin{cases} W_0[-\kappa_j, (1+\delta_a)/(1-\delta_a)], & \kappa_j \le 0; \\ W_0[\kappa_j, (1-\delta_a)/(1+\delta_a)], & \kappa_j > 0, \end{cases} \tag{23}$$

$$W_0(x,y) = \Psi(x,\infty,\infty,y) = (2y+1)\exp[y\,(y+1)\,|x|]\left\{1 - \Phi[(2y+1)\sqrt{|x|/4}]\right\} + \Phi[\sqrt{|x|/4}] - 1.$$

It is known [6, 8] that with the increase in SNR λ_{0j}, the quasi-liklelihood estimates converge in the rms sense to the locations of the maxima of mathematical expectations of the decision statistics given by Eqs. (5) and (6), which coincide with the true values θ_{0j} of the appearance and disappearance times for $|\delta_a| < 1$. Let us expand $(-1)^j\, Q(\theta,\theta_j)$ into its Taylor series in terms of the variable θ_j in the vicinity of θ_{0j} and confine ourselves to the first-order terms in the expansion:

$$(-1)^j\, Q(\theta,\theta_j) \approx (-1)^j\, Q(\theta_j,\theta_{0j}) + (-1)^j\, \rho_j^2\,(\theta_j - \theta_{0j})/T_{\max},$$

where $\rho_j^2 = 2a_0^2 f^2(\theta_{0j}) T_{\max}/N_0$ and $T_{\max} = \theta_{2\max} - \theta_{1\min}$ is the maximum possible signal duration. Hence, we obtain

$$\lambda_j - \lambda_{0j} \approx (-1)^j\, \rho_j^2\,(\theta_j - \theta_{0j})/T_{\max}. \tag{24}$$

Substituting Eq. (24) into Eq. (22), we obtain

$$\kappa_j = (-1)^j \times \begin{cases} (1-\delta_a^2)\,\rho_j^2\,(\theta_j - \theta_{0j})/T_{\max}, & \theta_j \le \theta_{0j}; \\ (1+\delta_a)^2\,\rho_j^2\,(\theta_j - \theta_{0j})/T_{\max}, & \theta_j > \theta_{0j}. \end{cases} \tag{25}$$

Using Eqs. (23) and (25), we find the asymptotic values of the bias and variance of the quasi-likelihood estimates of the appearance and disappearance times as

$$B_a(\theta_j^* \mid \theta_{0j}) = -\frac{(-1)^j\, 8T_{\max}\delta_a}{\rho_j^2\,(\delta_a - 1)^2\,(\delta_a + 1)^2}; \tag{26}$$

$$V_a(\theta_j^* \mid \theta_{0j}) = \frac{2T_{\max}^2\,(13 + 101\delta_a^2 + 15\delta_a^4 - \delta_a^6)}{\rho_j^4\,(\delta_a - 1)^4\,(\delta_a + 1)^4}. \tag{27}$$

For $\delta_a = 0$, the quantities given by Eqs. (26) and (27) coincide with the bias and variance of the maximum-likelihood estimate of the appearance and disappearance times of a signal with the *a priori* known amplitude, which were obtained in [4], namely,

$$B_{0j} = 0, \qquad V_{0j} = 26T_{\max}^2/\rho_j^4. \tag{28}$$

The influence of the *a priori* ignorance of the amplitude on the accuracy of the quasi-likelihood estimates of the appearance and disappearance times can be characterized by the normalized bias $b(\delta_a) = B_a(\theta_j^* \mid \theta_{0j})/\sqrt{V_a(\theta_j^* \mid \theta_{0j})}$ and the normalized variance $v(\delta_a) = V_a(\theta_j^* \mid \theta_{0j})/V_{0j}$. These quantities are the same for the quasi-likelihood estimates of the appearance and disappearance times and can characterize a loss in the accuracy of the quasi-likelihood estimates compared with the accuracy of the maximum-likelihood estimates of the appearance and disappearance times for a signal with known amplitude. Figures 1 and 2 show the normalized bias $b(\delta_a)$ and the normalized variance $v(\delta_a)$, respectively, as functions of δ_a. It is seen in Figs. 1 and 2 that the quasi-likelihood estimates of the appearance and disappearance times for the known amplitude ($\delta_a = 0$) have a zero bias, while their variance coincides with that of the maximum-likelihood estimates. The presence of the amplitude detuning results in pronounced deterioration of estimation quality. For example, for $|\delta_a| = 0.5$, the variance of the quasi-likelihood estimate is 10 times greater than that of the maximum-likelihood estimate in the case of an *a priori* known amplitude.

3. MAXIMUM-LIKELIHOOD ESTIMATION ALGORITHM

To improve the accuracy of estimating the appearance and disappearance times, one can use the maximum-likelihood algorithm according to which the unknown amplitude in Eq. (2) should be replaced by its estimate a_{m}, which is equivalent to minimization of the LRF logarithm in Eq. (2) with respect to the amplitude:

$$L(\theta_1, \theta_2) = L(a_{\mathrm{m}}, \theta_1, \theta_2) = \max_a L(a, \theta_1, \theta_2). \tag{29}$$

The maximum-likelihood estimates of the appearance and disappearance times are determined as the locations of the maximum of decision statistic (29):

$$(\theta_{\mathrm{m}1}, \theta_{\mathrm{m}2}) = \arg\sup L(\theta_1, \theta_2). \tag{30}$$

The LRF logarithm in Eq. (2) can analytically be maximized with respect to the amplitude. To this end, the derivative of the function in Eq. (2) with respect to a is put equal to zero

$$\left.\frac{\mathrm{d}L(a, \theta_1, \theta_2)}{\mathrm{d}a}\right|_{a_{\mathrm{m}}} = \frac{2}{N_0}\int_{\theta_1}^{\theta_2} \xi(t) f(t)\,\mathrm{d}t - \frac{2a_m}{N_0}\int_{\theta_1}^{\theta_2} f^2(t)\,\mathrm{d}t = 0.$$

Then we solve the obtained likelihood equation with respect to a_{m}:

$$a_{\mathrm{m}} = \int_{\theta_1}^{\theta_2} \xi(t) f(t)\,\mathrm{d}t \Bigg/ \int_{\theta_1}^{\theta_2} f^2(t)\,\mathrm{d}t. \tag{31}$$

Substituting the solution of Eq. (31) into Eq. (29), we obtain

$$L(\theta_1, \theta_2) = \frac{1}{N_0}\left(\int_{\theta_1}^{\theta_2} \xi(t) f(t)\,\mathrm{d}t\right)^2 \Bigg/ \int_{\theta_1}^{\theta_2} f^2(t)\,\mathrm{d}t. \tag{32}$$

Equation (32) determines the structure of the receiver which can be realized only in the multichannel option.

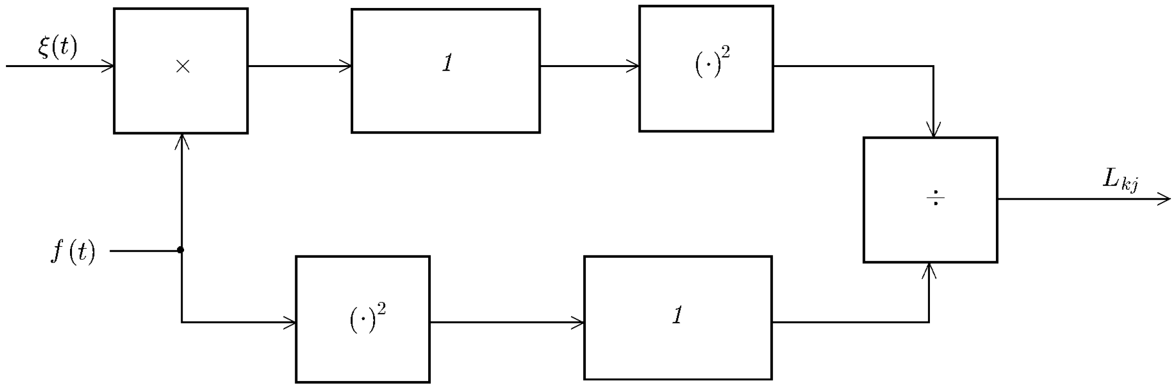

Fig. 3. Block diagram of one channel of a maximum-likelihood meter of the appearance and disappearance times.

The following values of decision statistic (32) are formed for a discrete set of values of the appearance and disappearance times: $L_{kj} = L(\theta_{1\,\min} + k\Delta\theta_1, \theta_{2\,\min} + j\Delta\theta_2)$, where $k = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$. Then the meter should consist of $n_1 n_2$ channels. A block diagram of one channel is shown in Fig. 3, where the labels *1* denote integrators in the time interval $[\theta_{1\,\min} + k\Delta\theta_1, \theta_{2\,\min} + j\Delta\theta_2]$. The of the maximum-likelihood estimates of the appearance and disappearance times are determined by the numbers of channels with the maximum output signal.

It should be noted that in addition to the difficulties of realization of the maximum-likelihood algorithm described by Eq. (30), which are due to its multichannel nature, one can face difficulties in determining the characteristics of estimate by the absolute maximum of the LRF logarithm given by Eq. (32).

4. QUASI-OPTIMAL ESTIMATION ALGORITHM

To simplify the hardware and software realization of the maximum-likelihood meter and finding its characteristics, one can use quasi-optimal estimates. In this case, we write the LRF logarithm in Eq. (2) as a sum $L(a, \theta_1, \theta_2) = L_1(a, \theta_1) + L_2(a, \theta_2)$ of the two terms

$$L_1(a, \theta_1) = \frac{2a}{N_0} \int\limits_{\theta_1}^{\theta} \xi(t) f(t)\,\mathrm{d}t - \frac{a^2}{N_0} \int\limits_{\theta_1}^{\theta} f^2(t)\,\mathrm{d}t; \tag{33}$$

$$L_2(a, \theta_2) = \frac{2a}{N_0} \int\limits_{\theta}^{\theta_2} \xi(t) f(t)\,\mathrm{d}t - \frac{a^2}{N_0} \int\limits_{\theta}^{\theta_2} f^2(t)\,\mathrm{d}t, \tag{34}$$

where θ is an arbitrary point which belongs to the interval $(\theta_{1\,\max}, \theta_{2\,\min})$. Denote $L_{aj}(\theta_j) = \max\limits_a L_j(a, \theta_j)$ and consider the estimates

$$\theta^*_{\mathrm{m}j} = \arg\sup L_{aj}(\theta_j). \tag{35}$$

Although the quasi-optimal estimates of Eq. (35) are not the maximum-likelihood estimates, it is shown below that with increasing SNR their efficiency asymptotically coincides with that of the maximum-likelihood estimates of the appearance and disappearance times for a signal with the *a priori* known amplitude. By analogy with Eqs. (31) and (32), we maximize the functions given by Eqs. (33) and (34) with respect to the variable a and obtain

$$L_{a1}(\theta_1) = \frac{1}{N_0} \left(\int\limits_{\theta_1}^{\theta} \xi(t) f(t)\,\mathrm{d}t \right)^2 \Bigg/ \int\limits_{\theta_1}^{\theta} f^2(t)\,\mathrm{d}t; \tag{36}$$

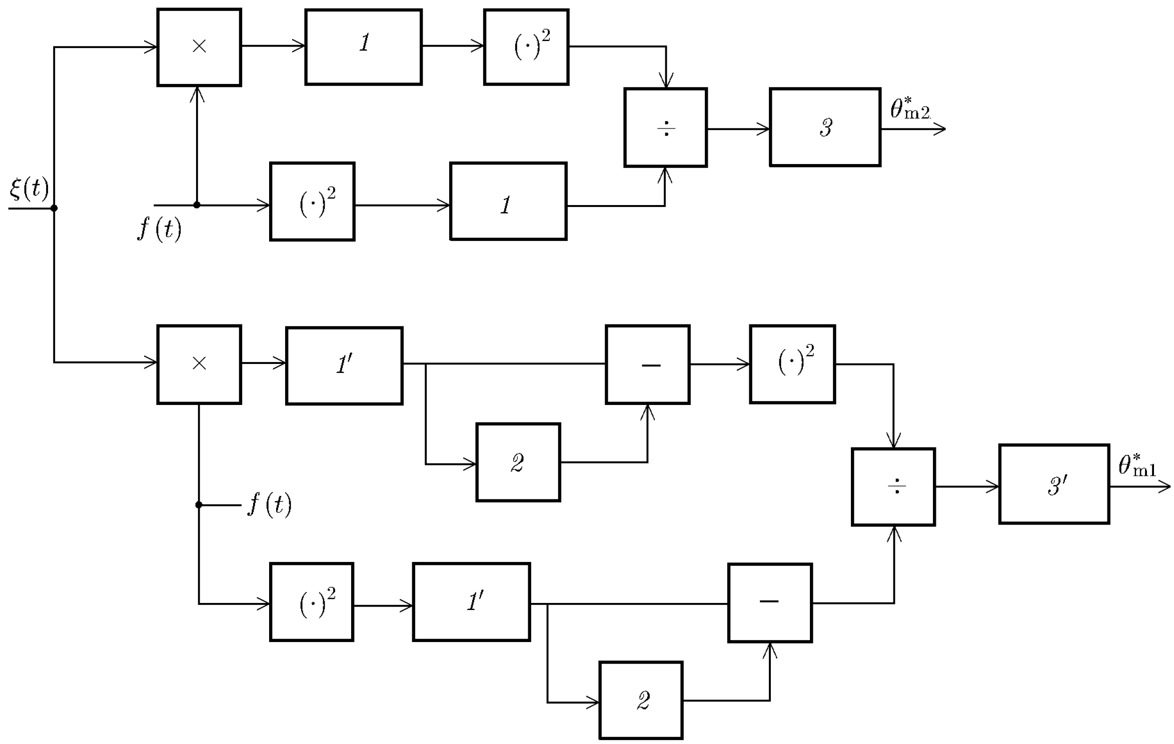

Fig. 4. Block diagram of the quasi-optimal meter of the appearance and disappearance times.

$$L_{a2}(\theta_2) = \frac{1}{N_0}\left(\int\limits_{\theta}^{\theta_2} \xi(t) f(t)\,\mathrm{d}t\right)^2 \Bigg/ \int\limits_{\theta}^{\theta_2} f^2(t)\,\mathrm{d}t. \tag{37}$$

Figure 4 shows a block diagram of the device for forming quasi-optimal estimates (35) of the appearance and disappearance times, which is developed on the basis of Eqs. (36) and (37). In Fig. 4, *1* and *1′* denote integrators in the time intervals $[\theta, t]$, where $t \in [\theta, \theta_{2\,\max}]$, and $[\theta_{1\,\min}, t]$, where $t \in [\theta_{1\,\min}, \theta]$, respectively, *2* denotes the delay line for the time $\Delta t = \theta - \theta_{1\,\min}$, and *3* and *3′* stand for solvers that search for the signal-maximum location in the time intervals $[\theta_{2\,\min}, \theta_{2\,\max}]$ and $[\theta, \theta + \theta_{1\,\max} - \theta_{1\,\min}]$, respectively. Therefore, using the estimates given by Eq. (35) one can significantly simplify the technical realization of the receiver. Indeed, to realize the maximum-likelihood estimation algorithm given by Eq. (29), we should develop a multichannel receiver in terms of the appearance and disappearance times. For finding the estimates given by Eq. (35), a two-channel circuit is sufficient.

Now let us analyze estimation algorithm (35). Consider the random processes

$$M_1(\theta_1) = \int\limits_{\theta_1}^{\theta} \xi(t) f(t)\,\mathrm{d}t, \qquad M_2(\theta_2) = \int\limits_{\theta}^{\theta_2} \xi(t) f(t)\,\mathrm{d}t, \tag{38}$$

which are squared in Eqs. (36) and (37). They are the Gaussian random processes with the mathematical expectations

$$S_1(\theta_1) = \langle M_1(\theta_1)\rangle = a_0 q[\max(\theta_{01}, \theta_1), \theta], \qquad S_2(\theta_2) = \langle M_2(\theta_2)\rangle = a_0 q[\theta, \min(\theta_{02}, \theta_2)]$$

and the correlation functions

$$B_1(\theta_{11}, \theta_{21}) = \langle [M_1(\theta_{11}) - S_1(\theta_{11})]\,[M_1(\theta_{21}) - S_1(\theta_{21})]\rangle = N_0 q[\max(\theta_{11}, \theta_{21}), \theta]/2,$$

$$B_2(\theta_{12}, \theta_{22}) = \langle [M_2(\theta_{12}) - S_2(\theta_{12})]\, [M_2(\theta_{22}) - S_2(\theta_{22})] \rangle = N_0 q[\theta, \min(\theta_{12}, \theta_{22})]/2,$$

where

$$q(\theta_1, \theta_2) = \int\limits_{\theta_1}^{\theta_2} f^2(t)\, \mathrm{d}t. \tag{39}$$

Since the function $f(t)$ can turn to zero only in a zero-measure part of the interval $[\theta_{1\,\mathrm{min}}, \theta_{2\,\mathrm{max}}]$, $q(\theta_1, \theta)$ is a monotonically decreasing function of the argument θ_1, and $q(\theta, \theta_2)$ is a monotonically increasing function of θ_2, and the equalities $q(x, \theta) = -q(\theta, x)$,

$$q[\max(x, y), \theta] = \min[q(x, \theta), q(y, \theta)], \qquad q[\theta, \min(x, y)] = \min[q(\theta, x), q(\theta, y)]$$

take place. Using the properties of function (39), one can rewrite the mathematical expectations and the correlation functions of the processes given by Eq. (38) in the form

$$S_j(\theta_j) = a_0 \min[(-1)^j\, q(\theta, \theta_{0j}), (-1)^j\, q(\theta, \theta_j)],$$
$$B_j(\theta_{1j}, \theta_{2j}) = N_0 \min[(-1)^j\, q(\theta, \theta_{1j}), (-1)^j\, q(\theta, \theta_{2j})]/2.$$

In Eq. (38), we replace the variables θ_1 and θ_2 by new variables $r_j = (-1)^j\, q(\theta, \theta_j)$ such that $r_j \in [R_{j\,\mathrm{min}}, R_{j\,\mathrm{max}}]$, $R_{1\,\mathrm{min}} = q(\theta_{1\,\mathrm{max}}, \theta)$, $R_{1\,\mathrm{max}} = q(\theta_{1\,\mathrm{min}}, \theta)$, $R_{2\,\mathrm{min}} = q(\theta, \theta_{2\,\mathrm{min}})$, and $R_{2\,\mathrm{max}} = q(\theta, \theta_{2\,\mathrm{max}})$. Then for the random processes in Eqs. (36) and (37), we can write

$$L_{aj}(\theta_j) = (-1)^j\, M_j^2(\theta_j)/[N_0 q(\theta, \theta_j)] = L_{aj}(r_j) = \chi_j^2(r_j)/(N_0 r_j).$$

Here, $\chi_j(r_j)$ are the statistically independent Gaussian random processes with the mathematical expectations $S_j(r_j) = a_0 \min(r_j, r_{0j})$, where $r_{0j} = (-1)^j\, q(\theta, \theta_{0j})$ and the correlation functions $B_j(r_{1j}, r_{2j}) = N_0 \min(r_{1j}, r_{2j})/2$. Therefore, for decision statistics (36) and (37), the following expression holds true:

$$L_{aj}(l_j) = z_{0j}^2\, [\min(1, l_j)]^2/(2l_j) + z_{0j} \min(1, l_j)\omega_j(l_j)/l_j + \omega_j^2(l_j)/(2l_j), \tag{40}$$

where

$$l_j = r_j/r_{0j}, \qquad l_j \in [L_{j\,\mathrm{min}}, L_{j\,\mathrm{max}}], \qquad L_{j\,\mathrm{min}} = R_{j\,\mathrm{min}}/r_{0j}, \qquad L_{j\,\mathrm{max}} = R_{j\,\mathrm{max}}/r_{0j},$$
$$z_{0j}^2 = 2a_0^2 r_{0j}/N_0 = 2a_0^2\, (-1)^j\, q(\theta, \theta_{0j})/N_0,$$

and $\omega_j(l_j)$ are the statistically independent standard Wiener processes.

For large SNRs z_{0j}, the last term in Eq. (40) can be ignored and approximately written as

$$L_{aj}(l_j) \approx z_{0j}^2\, [\min(1, l_j)]^2/(2l_j) + z_{0j} \min(1, l_j)\omega_j(l_j)/l_j. \tag{41}$$

These functions are the Gaussian random processes with the mathematical expectations

$$S_{aj}(l_j) = z_{0j}^2\, [\min(1, l_j)]^2/(2l_j) \tag{42}$$

and the correlation functions

$$K_j(l_{1j}, l_{2j}) = z_{0j}^2 \min(l_{1j}, 1) \min(l_{2j}, 1) \min(l_{1j}, l_{2j})/(l_{1j} l_{2j}). \tag{43}$$

For large SNRs z_{0j}, the maximum of the decision-statistic is located in the immediate vicinity of its mathematical-expectation maximum [6]. The mathematical expectations given by Eq. (42) attain maximum values for $l_j = 1$. Therefore, we study the behavior of random processes (41) in the immediate vicinities of the points $l_j = 1$. To this end, expanding the functions in Eqs. (42) and (43) into the Taylor series in terms of l_j and (l_{1j}, l_{2j}) in the vicinity of unity, we obtain

$$S_{aj}(l_j) \approx z_{0j}^2 \,(1 - |l_j - 1|)/2, \tag{44}$$
$$K_j(l_{1j}, l_{2j}) \approx z_{0j}^2 \,(1 - |l_{1j} - 1|/2 - |l_{2j} - 1|/2 - |l_{1j} - l_{2j}|/2). \tag{45}$$

For large SNRs, we approximate decision statistic (41) by the Gaussian random processes $Y_j(l_j)$ with mathematical expectations (44) and correlation functions (45) in the intervals of the possible values of the variables l_j. Locations of the maxima of the random processes $Y_j(l_j)$, i.e.,

$$l_{\mathrm{m}j} = \arg\sup Y_j(l_j), \tag{46}$$

are related to the estimates of the appearance and disappearance times by one-to-one transformations. Therefore, the distribution functions of the estimates given by Eq. (35) can be obtained using the distribution functions of the random quantities given by Eq. (46):

$$F_j(x) = \{l_{\mathrm{m}j} < x\} = P\left[\sup_{l_j \le x} Y_j(l_j) > \sup_{l_j > x} Y_j(l_j)\right]. \tag{47}$$

Let us introduce the random processes

$$\eta_j(l_j) = \left[Y_j(l_j) - Y_j(x)\right]/z_{0j}, \qquad x \in [L_{j\,\min}, L_{j\,\max}], \tag{48}$$

which make it possible to rewrite Eq. (47) as

$$F_j(x) = P\left[\sup_{l_j \le x} \eta_j(l_j) > \sup_{l_j > x} \eta_j(l_j)\right]. \tag{49}$$

By definition, the quantities $\eta_j(l_j)$ in Eq. (48) are the statistically independent Gaussian random processes with the mathematical expectations

$$S_j(l_j) = z_{0j}(|x - 1| - |l_j - 1|)/2$$

and the correlation functions

$$K_j(l_{1j}, l_{2j}) = \begin{cases} \min(|l_{1j} - x|, |l_{2j} - x|), & (l_{1j} - x)\,(l_{2j} - x) \ge 0; \\ 0, & (l_{1j} - x)\,(l_{2j} - x) < 0. \end{cases} \tag{50}$$

According to Eq. (50), the realization segments of the random processes $\eta_j(l_j)$ in the intervals $[L_{j\,\min}, x]$ and $(x, L_{j\,\max}]$ are statistically independent. Therefore, by analogy with [10], we can write the following expression for the distribution given by Eqs. (47) and (49):

$$F_j(x) = \int_0^\infty P_{2j}(u)\,\mathrm{d}P_{1j}(u), \tag{51}$$

where

$$P_{1j}(u) = P\left[\sup_{l_j \le x} \eta_j(l_j) < u\right], \qquad P_{2j}(v) = P\left[\sup_{l_j > x} \eta_j(l_j) < v\right].$$

The random processes $\eta_j(l_j)$ are the statistically independent Gaussian Markov processes [9] with the drift and diffusion coefficients

$$k_{1j} = z_{0j} \times \begin{cases} 1/2, & L_{j\,\min} \le l_j \le 1, \\ -1/2, & 1 < l_j \le L_{j\,\max}, \end{cases} \qquad k_{2j} = 1. \tag{52}$$

Therefore, the functions $P_{1j}(u)$ are the probabilities of that the boundary u is not reached by the Markov random processes $\eta_j(l_j)$ for $L_{j\,\min} \le l_j \le 1$, while the functions $P_{2j}(v)$ are the probabilities that the boundary v is not reached by the Markov random processes $\eta_j(l_j)$ for $1 < l_j \le L_{j\,\max}$.

By analogy with [9], we have

$$P_{1j}(u) = \int_0^u W_{1j}(y, L_{j\,\min})\,\mathrm{d}y, \qquad P_{2j}(v) = \int_0^v W_{2j}(y, L_{j\,\max})\,\mathrm{d}y, \tag{53}$$

where $W_{2j}(y, l_j)$ are the solutions of the direct Fokker–Planck–Kolmogorov equation (17) with the coefficients given by Eq. (52) for the boundary conditions $W_{2j}(y = -\infty, l_j) = W_{2j}(y = u, l_j) = 0$ and the initial condition $W_{2j}(y, l_j = x) = \delta(y - u)$, while $W_{1j}(y, l_j)$ are the solutions of the inverse Fokker–Planck–Kolmogorov equation

$$\frac{\partial W_{1j}(y, l_j)}{\partial l_j} + k_{1j}\frac{\partial}{\partial y}[W_{1j}(y, l_j)] + \frac{k_{2j}}{2}\frac{\partial^2}{\partial y^2}[W_{1j}(y, l_j)] = 0 \tag{54}$$

with the coefficients given by Eq. (52) for the boundary conditions $W_{1j}(y = -\infty, l_j) = W_{1j}(y = v, l_j) = 0$ and the initial condition $W_{1j}(y, l_j = x) = \delta(y - v)$. Solving Eqs. (17) and (54) by the reflection method with the sign reversal [9], substituting the obtained solutions into Eq. (53), and then substituting Eq. (53) into Eq. (51), we find the distribution function of the random quantity in Eq. (46):

$$F_j(x) = \begin{cases} P\left[\dfrac{z_{0j}^2}{4}(x - L_{j\,\min}), \dfrac{z_{0j}^2}{4}(1 - x), \dfrac{z_{0j}^2}{4}(L_{j\,\max} - 1)\right], & L_{j\,\min} \le x \le 1; \\ 1 - P\left[\dfrac{z_{0j}^2}{4}(L_{j\,\max} - 1), \dfrac{z_{0j}^2}{4}(x - 1), \dfrac{z_{0j}^2}{4}(x - L_{j\,\min})\right], & 1 < x \le L_{j\,\max}. \end{cases} \tag{55}$$

Here, we denote

$$\begin{aligned} P(x_1, x_2, x_3) = {} & \frac{1}{2\sqrt{2\pi x_2}} \int_0^\infty \int_0^\infty \left\{ \exp\left[-\frac{(\xi - u)^2}{8x_2}\right] - \exp\left[-\frac{(\xi + u)^2}{8x_2}\right] \right\} \\ & \times \left\{ \exp(-u)\Phi\left(\frac{2x_1 - u}{2\sqrt{x_1}}\right) + \frac{1}{\sqrt{2\pi x_1}} \exp\left[-\frac{(2x_1 + u)^2}{8x_1}\right] \right\} \exp\left[-\frac{\xi - u}{2} - \frac{x_2}{2}\right] \\ & \times \left\{ \Phi\left(\sqrt{x_3} + \frac{\xi}{2\sqrt{x_3}}\right) - \exp(-\xi)\Phi\left(\sqrt{x_3} - \frac{\xi}{2\sqrt{x_3}}\right) \right\} \mathrm{d}\xi\,\mathrm{d}u. \end{aligned}$$

Let us consider the asymptotic behavior of the distribution function in Eq. (55) with increasing SNR. Assuming that the true values of the appearance and disappearance times are the internal points of their *a priori* intervals and letting $z_{0j} \to \infty$, we find the limiting expression for the distribution function given by Eq. (55):

$$F_{0j}(x) = \begin{cases} P_0\left[z_{0j}^2(1 - x)/4\right], & L_{j\,\min} \le x \le 1, \\ 1 - P_0\left[z_{0j}^2(x - 1)/4\right], & 1 < x \le L_{j\,\max}. \end{cases}$$

where

$$P_0(x) = P(+\infty, x, +\infty) = \int\limits_0^\infty \exp(-u) \left[\Phi\left(\frac{u-2x}{2\sqrt{x}}\right) - \exp(u)\Phi\left(-\frac{u+2x}{2\sqrt{x}}\right) - \exp(-u+4x)\Phi\left(\frac{u-6x}{2\sqrt{x}}\right) + \exp(2u+4x)\Phi\left(-\frac{u+6x}{2\sqrt{x}}\right)\right] \mathrm{d}u.$$

We also find the limiting probability density of the random quantities given by Eq. (46)

$$W_{0j}(x) = \frac{3z_{0j}^2}{2} \exp\left[z_{0j}^2 \, |x-1|\right] \left\{1 - \Phi\left(\frac{3z_{0j}}{2}\sqrt{|x-1|}\right)\right\} - \frac{z_{0j}^2}{2}\left\{1 - \Phi\left(\frac{z_{0j}}{2}\sqrt{|x-1|}\right)\right\}. \tag{56}$$

Let us expand the expression $(-1)^j \, q(\theta, \theta_j)$ into a Taylor series in terms of the variable θ_j in the vicinity of θ_{0j} and confine ourselves to the first-order terms in the series:

$$(-1)^j \, q(\theta, \theta_j) \approx (-1)^j \, q(\theta, \theta_{0j}) + (-1)^j \frac{\rho_j^2 N_0}{2a_0^2 T_{\max}} (\theta_j - \theta_{0j}),$$

where $\rho_j^2 = 2f^2(\theta_{0j}) a_0^2 T_{\max}/N_0$. Hence, we obtain

$$l_{\mathrm{m}j} - 1 \approx (-1)^j \frac{\rho_j^2}{T_{\max} z_{0j}^2} (\theta_j - \theta_{0j}). \tag{57}$$

Using Eqs. (57) and (56), we find asymptotic values of the bias and variance of the quasi-optimal appearance and disappearance times given by Eq. (35):

$$B_a(\theta_{mj}^* \mid \theta_{0j}) = 0, \qquad V_a(\theta_{mj}^* \mid \theta_{0j}) = \frac{26 T_{\max}^2}{\rho_j^4}, \tag{58}$$

which coincide with those in Eq. (28). It is evident from Eq. (58) that with increasing SNR the accuracy of the quasi-optimal appearance and disappearance times in Eq. (35) asymptotically coincides with that of the maximum-likelihood estimates of the appearance and disappearance times for a signal with the *a priori* known amplitude. Therefore, the estimates of Eq. (35) are asymptotically maximum-likely. Using them, we can significantly simplify technical realization of the maximum likelihood meter of the appearance and disappearance times given by Eq. (30).

Therefore, the asymptotic values of the variances of the maximum-likelihood estimates of the appearance and disappearance times for a signal with the *a priori* known amplitude, which are given by Eq. (28), and the asymptotic value of the variances of the quasi-optimal estimates, which are given in Eq. (58), coincide. Hence it follows that the asymptotic values of the variances of maximum-likelihood estimates (30) of the appearance and disappearance times for a signal with the *a priori* unknown amplitude are also determined by Eqs. (28) and (58). Indeed, the variance of estimates (30) of the appearance and disappearance times for a signal with the *a priori* unknown amplitude cannot be smaller than the variance in Eq. (28) for the maximum-likelihood estimates of the appearance and disappearance times for a signal with the *a priori* known amplitude. At the same time, the variance of the maximum-likelihood estimates given by Eq. (30) cannot be greater than the variance in Eq. (58) for the quasi-optimal estimates.

Therefore, the dependences shown in Figs. 1 and 2 characterize an accuracy gain for the maximum-likelihood estimates given by Eq. (30) or the quasi-optimal estimates given by Eq. (35) compared with the quasi-likelihood estimates given by Eq. (4). Therefore, for $|\delta_a| \geq 1/2$, the variance of the maximum-likelihood and quasi-optimal estimates, which are given by Eqs. (30) and (35), respectively, is smaller by more than an order of magnitude than that of the quasi-likelihood estimates given by Eq. (4).

5. COMPUTER STATISTICAL SIMULATION

To check the efficiency of the synthesized estimation algorithms and establish the applicability limits of the asymptotic expressions for their effectiveness characteristics, the computer-aided statistical simulation of the estimates given by Eqs. (30) and (35) was performed using a tapered rectangular pulse as an example [11]. The signal shape is described by the function

$$f(t) = (1 + bt/T_{\max})\,(1 + b + b^2/3)^{-1/2}, \tag{59}$$

where the quantity b characterizes the pulse-top tilt. The factor $(1 + b + b^2/3)^{-1/2}$ is introduced to Eq. (59) to ensure that the energy of the maximum-duration signal independent of the pulse-top tilt. This allows one to compare the efficiency of estimating the appearance and disappearance times for signals with different top tilts but the same energy.

The *a priori* regions of possible values of the appearance and disappearance times are chosen such that $\theta_{1\,\min} = 0$ and $\theta_{2\,\max} = T_{\max}$ are fixed and the maximum signal duration $T_{\max}$ remains intact. We choose $\theta = T_{\max}/2$ as the center of the interval $[0, T_{\max}]$. Assume that the points $\theta_{1\,\max}$ and $\theta_{2\,\min}$ are located symmetrically with respect to θ and their locations can vary only in accordance with the variation in $k = T_{\max}/T_{\min}$, where $T_{\min} = \theta_{2\,\min} - \theta_{1\,\max}$ is the minimum signal duration. Assume that the true values of the appearance and disappearance times are located at the center of their *a priori* intervals $\theta_{01} = T_{\max}\,(k-1)/(4k)$ and $\theta_{02} = T_{\max}\,(3k+1)/(4k)$, respectively.

The maximum-likelihood and quasi-optimal estimates were simulated. During simulation of the maximum-likelihood estimates given by Eq. (30), the discrete readouts of decision statistic (32) were formed:

$$L_{ij} = \frac{\left\{ zS(\max[\xi_{1i}, \xi_{01}], \min[\xi_{2j}, \xi_{02}]) + \sum\limits_{n=i}^{j} [x_n \sqrt{\Delta\xi}(1 + bn\,\Delta\xi)] \Big/ \sqrt{1 + b + b^2/3} \right\}^2}{2S(\xi_{1i}, \xi_{2j})}, \tag{60}$$

$$S(x, y) = \left[y - x + b\,(y^2 - x^2) + b^2\,(y^3 - x^3)/3\right] / (1 + b + b^2/3)\ ,$$

where $\Delta\xi = 10^{-5}$ is the discretization step of the normalized time $\xi = t/T_{\max}$, x_n are the statistically independent Gaussian random quantities with zero mathematical expectations and unit variances, $\xi_{1i} = \theta_{1i}/T_{\max} = i\,\Delta\xi$ and $\xi_{2j} = \theta_{2j}/T_{\max} = j\,\Delta\xi$ are the discretization nodes of the appearance and disappearance times, respectively, $i = 0, 1, \ldots, \mathrm{ent}[(k-1)/(2k\,\Delta\xi)]$, $j = \mathrm{ent}[(k+1)/(2k\,\Delta\xi)], \ldots, N-1, N$, $N = \mathrm{ent}(1/\Delta\xi)$, $\mathrm{ent}(x)$ is the integer part of the number x, $\xi_{01} = \theta_{01}/T_{\max}$, $\xi_{02} = \theta_{02}/T_{\max}$, and $z^2 = 2T_{\max}a_0^2/N_0$ is the SNR at the output of the maximum-likelihood receiver for a rectangular pulse. On the basis of the readouts given by Eq. (60) the maximum-likelihood estimates $\theta_{m1} = T_{\max}\,\Delta\xi\, i_m$ and $\theta_{m2} = T_{\max}\,\Delta\xi\, j_m$ of the appearance and disappearance times, respectively, were formed, where i_m and j_m are the numbers of the maximum readout of the decision statistic.

In the simulation of quasi-optimal estimates (35), the following discrete readouts of the random processes, given by Eqs. (33) and (34), were formed:

$$L_{1i} = \frac{\left\{ zS(\max[\xi_{1i}, \xi_{01}], 1/2) + \sum\limits_{n=i}^{N/2} [x_n \sqrt{\Delta\xi}\,(1 + bn\,\Delta\xi)] \Big/ \sqrt{1 + b + b^2/3} \right\}^2}{2S(1/2, \xi_{1i})}, \tag{61}$$

$$L_{2j} = \frac{\left\{ zS(1/2, \min[\xi_{2i}, \xi_{02}]) + \sum\limits_{n=N/2}^{j} [x_n \sqrt{\Delta\xi}\,(1 + bn\,\Delta\xi)] \Big/ \sqrt{1 + b + b^2/3} \right\}^2}{2S(\xi_{2i}, 1/2)}. \tag{62}$$

On the basis of the readouts in Eqs. (61) and (62), the following quasi-optimal estimates of the appearance and disappearance times were developed: $\theta^*_{m1} = T_{\max}\,\Delta\xi\, i^*_m$ and $\theta^*_{m2} = T_{\max}\,\Delta\xi\, j^*_m$, where i^*_m and j^*_m are the numbers of the maximum readouts given by Eqs. (61) and (62), respectively.

During the simulation, 10^5 test cycles were realized for each z. Therefore, boundaries of the confidence intervals deviate from the experimental values of variances by no more than 15% with a probability of 0.9 for $V(\theta_i \mid \theta_{0i})/T^2_{\max} > 10^{-3}$.

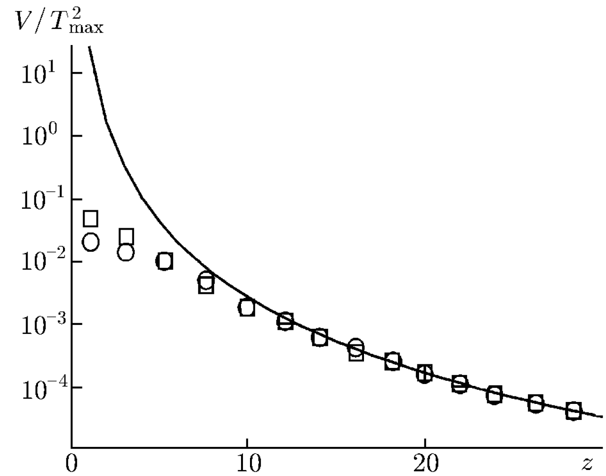

Fig. 5. Variances of the disappearance-time estimates.

Figure 5 shows the variance (normalized to $T^2_{\max}$) the disappearance-time estimates for signal (59) as a function of the SNR z for $b = 0$ and $k = 10$. The solid curve shows the asymptotic dependence calculated by Eqs. (28) and (58), the circles show the experimental values of the variance of the maximum-likelihood estimate given by Eq. (30), and the squares denote experimental values of the variance of the quasi-optimal estimate given by Eq. (35), which were obtained during the simulation. As is evident from Fig. 5, the scattering of the maximum-likelihood estimate is somewhat smaller than that of the quasi-optimal estimate if SNRs are not too large. The difference between them decreases with increasing SNR, and the variances of the maximum-likelihood and quasi-optimal estimates coincides with their asymptotic value.

6. CONCLUSIONS

We have synthesized the quasi-likelihood, maximum-likelihood, and quasi-optimal algorithms for estimating the appearance and disappearance times of an arbitrary-shaped signal with unknown amplitude. The asymptotic characteristics of the operation quality of the synthesized algorithms were found. Although the maximum-likelihood algorithm has the best estimation accuracy compared with other considered algorithms, its hardware or software realization is most complicated. The accuracy of a simpler quasi-optimal estimation algorithm asymptotically coincides with that of the maximum-likelihood algorithm. It was shown that the absence of *a priori* information on the signal amplitude for large SNRs does not asymptotically influence the accuracy of the maximum-likelihood and quasi-optimal estimates of the signal appearance and disappearance times. The obtained results allow us to reasonably choose the estimation algorithm depending on the available *a priori* information on the signal amplitude and the requirements to the algorithm estimation accuracy and realization simplicity.

REFERENCES

1. V. G. Repin, *Probl. Inf. Trans.*, **27**, No. 1, 49 (1991).
2. A. A. Mal'tsev and A. M. Silaev, *Radiotekh. Élektron.*, **34**, No. 5, 1024 (1989).
3. A. P. Trifonov and Yu. É. Korchagin, *Radiophys. Quantum Electron.*, **63**, No. 3, 245 (2000).
4. A. P. Trifonov and Yu. É. Korchagin, *Probl. Inf. Trans.*, **37**, No. 1, 46 (2001).
5. A. P. Trifonov and Yu. É. Korchagin, *Probl. Inf. Trans.*, **45**, No. 2, 158 (2009).
6. E. I. Kulikov and A. P. Trifonov, *Signal Parameter Estimation against a Noise Background* [in Russian], Sovetskoe Radio, Moscow (1978).
7. V. I. Mudrov and V. L. Kushko, *Measurement Processing Methods* [in Russian], Radio i Svyaz', Moscow (1983).

8. A. P. Trifonov and Yu. S. Shinakov, *Joint Discrimination of Signals and Estimation of their Parameters against a Noise Background* [in Russian], Radio i Svyaz', Moscow (1986).

9. V. I. Tikhonov and M. A. Mironov, *Markov Processes* [in Russian], Sovetskoe Radio, Moscow (1977).

10. A. S. Terent'ev, *Radiotekh. Élektron.*, **13**, No. 4, 652 (1968).

11. M. I. Gryaznov, M. A. Gurevich, and M. A. Ryabinin, *Pulse Parameter Measurements* [in Russian], Radio i Svyaz', Moscow (1991).