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Amplitude Estimation of Signal with Unknown Duration

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Abstract—Quasilikelihood and maximum likelihood algorithms for estimating the amplitude of arbitrary waveform signal with unknown duration have been synthesized. Characteristics of the synthesized algorithms have been also found.

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The problem of amplitude estimation of a signal with unknown parameters was discussed in a number of papers [1–3] and others. This problem is urgent for many applications in the field of radio location and communication. Algorithms of amplitude estimation of signals with unknown non-power parameters were investigated in papers [1, 2], while the algorithms of amplitude estimation of signal with rectangular waveform and unknown duration were investigated in paper [3]. However, the real conditions of generation and propagation lead to the need of estimating the amplitude of signals having the waveform different from rectangular. The synthesis and analysis of the quasilikelihood and maximum likelihood algorithms for estimating the amplitudes of signals with arbitrary waveform and unknown duration are presented below.

Let the following realization of additive mixture of the useful signal and Gaussian white noise be accessible for observation on time interval [0, T]:

$$\xi(t) = s(t, a_0, \tau_0) + n(t),$$

where $s(t, a_0, \tau_0)$ is the useful signal,

$$s(t, a_0, \tau_0) = \begin{cases} a_0 f(t), & 0 \le t \le \tau_0, \\ 0, & t < 0, t > \tau_0, \end{cases} \tag{1}$$

$n(t)$ is the Gaussian white noise with one-sided spectral density N_0. Here a_0 and τ_0 are the received signal unknown amplitude and duration, respectively, $f(t)$ is the function describing the signal waveform. We shall assume that the signal duration takes on values from a priori interval $\tau \in [T_1, T_2]$. Given observable realization $\xi(t)$, it is necessary to generate an estimate of amplitude a_0 of useful signal (1).

For synthesizing an algorithm of amplitude estimation we shall employ the maximum likelihood (ML) method [1, 2], according to which the amplitude estimate coincides with the position of the absolute (largest) maximum of the logarithm of likelihood ratio functional (LRF). However, in case both the duration and amplitude are unknown, the LRF logarithm depends on two unknown parameters [1, 2]:

$$L(a, \tau) = \frac{2a}{N_0} \int_0^{\tau} \xi(t) f(t) \mathrm{d}t - \frac{a^2}{N_0} \int_0^{\tau} f^2(t) \mathrm{d}t. \tag{2}$$

Thus, there is an a priori parametric uncertainty in respect of the signal duration. One of the techniques for overcoming this uncertainty implies the application of quasilikelihood (QL) algorithm of estimation [4]. A quasilikelihood receiver generates the LRF logarithm (2) for certain expected duration τ^*

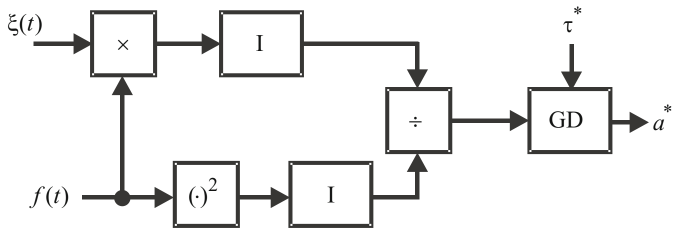

Fig. 1.

$$L^*(a) = L(a,\tau^*) = \frac{2a}{N_0}\int_0^{\tau^*}\xi(t)f(t)\mathrm{d}t - \frac{a^2}{N_0}\int_0^{\tau^*}f^2(t)\mathrm{d}t \tag{3}$$

and finds the QL estimate of amplitude as a position of the absolute maximum of the decision statistics (3)

$$a^* = \arg\sup L^*(a). \tag{4}$$

Expressions (3) and (4) determine the structure of the receiving device. The signal amplitude estimate can be found analytically. To this end, we shall equate to zero the derivative of function (3) with respect to a:

$$\left.\frac{\partial L_2(a,\tau)}{\partial a}\right|_{a^*} = \frac{2}{N_0}\int_0^{\tau^*}\xi(t)f(t)\mathrm{d}t - \frac{2a^*}{N_0}\int_0^{\tau^*}f^2(t)\mathrm{d}t = 0$$

and solve the resultant likelihood equation for a^*

$$a^* = \frac{\int_0^{\tau^*}\xi(t)f(t)\mathrm{d}t}{\int_0^{\tau^*}f^2(t)\mathrm{d}t}. \tag{5}$$

Figure 1 displays a block diagram of the QL meter of amplitude (5). The following designations are used in this diagram: I are the integrators performing integration on time interval $[0, t]$, $t \in [0; T_2]$, GD is the gating device performing the sampling of a value of amplitude estimate a^* at the instant of time τ^*.

Now we shall perform the analysis of the QL algorithm of amplitude estimate (4). To this end, we shall find the mathematical expectation of quantity (5):

$$M_a = \langle a^*\rangle = a_0\Big[1 - \eta(\tau^* - \tau_0) + \eta(\tau^* - \tau_0)/(1+\delta_\tau)\Big], \tag{6}$$

where

$$\eta(x) = \begin{cases} 1, x > 0, \\ 0, x \le 0, \end{cases}$$

$$\delta_\tau = (q(\tau^*) - q(\tau_0))/q(\tau_0), \tag{7}$$

$q(\tau)=\dfrac{2a_0^2}{N_0}\int_0^\tau f^2(t)\mathrm{d}t$ is the signal-to-noise ratio (SNR) for signal having duration τ. Quantity δ_τ (7) shall be called "generalized detuning" in terms of duration. It characterizes the relative difference of the energy of signal with expected duration τ^* with respect to the energy of signal with true value of duration τ_0. In accordance with relationship (6) the QL estimate of amplitude has a bias

$$B_a=\langle a^* - a_0\rangle = -a_0\delta_\tau\eta(\tau^* - \tau_0)/(1+\delta_\tau). \tag{8}$$

Performing the averaging we obtain expressions for the dispersion and scattering of estimate (5):

$$D_a=\left\langle (a_m^* - \langle a_m^*\rangle)^2\right\rangle = a_0^2/z_0^2(1+\delta_\tau), \tag{9}$$

$$V_a=\left\langle (a_m^* - a_0)^2\right\rangle = a_0^2/z_0^2(1+\delta_\tau)+a_0^2\delta_\tau^2\eta(\tau^*-\tau_0)/(1+\delta_\tau)^2, \tag{10}$$

where $z_0^2=q(\tau_0)=\dfrac{2a_0^2}{N_0}\int_0^{\tau_0} f^2(t)\mathrm{d}t$ is SNR for the received signal.

If the expected duration coincides with its true value, i.e., $\tau^*=\tau_0$, the QL estimate coincides with the ML estimate of amplitude provided the duration is a priori known [1]:

$$a_{0m}=\int_0^{\tau_0}\xi(t)f(t)\mathrm{d}t \bigg/ \int_0^{\tau_0} f^2(t)\mathrm{d}t. \tag{11}$$

The bias and scattering of estimate (11) were determined in paper [1]:

$$B=\langle a_{0m}-a_0\rangle = 0,$$

$$V=\langle (a_{0m}-a_0)^2\rangle = a_0^2/z_0^2. \tag{12}$$

These results coincide with expressions (8) and (10) at $\delta_\tau=0$.

Let us introduce into consideration quantity $\chi=V_a/V$ that characterizes the rise of scattering of the amplitude QL estimate in case of the unknown duration as compared with the scattering of the amplitude ML estimate in case of the known duration of signal:

$$\chi=V_a/V=1/(1+\delta_\tau)+z_0^2\delta_\tau^2\eta(\tau^*-\tau_0)/(1+\delta_\tau)^2. \tag{13}$$

As can be seen from expression (13), if expected duration τ^* of signal is less than the true value of duration τ_0, the loss χ in accuracy of the QL estimate as compared with the accuracy of ML estimate does not depend on SNR for the received signal. At $\tau^*>\tau_0$ the loss in accuracy of the QL estimate increases with the rise of SNR.

As an example let us consider the amplitude estimate characteristics of signal having the waveform of a rectangle with beveled vertex [5]

$$f(t)=\frac{1+bt/T_2}{\sqrt{1+b+b^2/3}}, \tag{14}$$

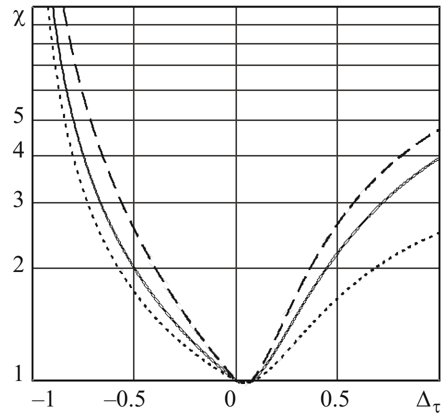

Fig. 2.

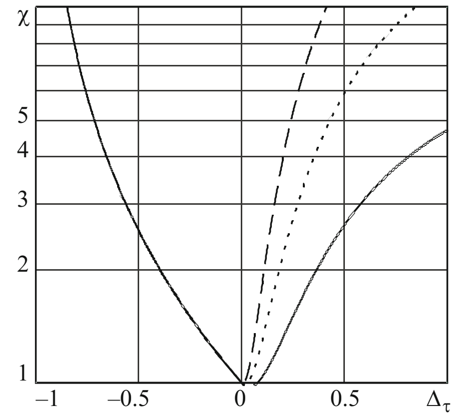

Fig. 3.

where quantity b characterizes the tilt of beveled top. Factor $(1+b+b^2/3)^{-1/2}$ is introduced for the purpose of making the energy of the maximum duration signal not dependent on the tilt of pulse peak making it possible to compare the amplitude estimation accuracy of signals with different tilts of the peak, but equal energies.

The generalized detuning (7) for signal (14) assumes the form

$$\delta_\tau = \frac{(1+\Delta_\tau)(1+b(1+\Delta_\tau)T_0 + b^2(1+\Delta_\tau)^2 T_0^2/3)}{1+bT_0+b^2T_0^2/3} - 1, \tag{15}$$

where $T_0 = \tau_0/T_2$, $\Delta_\tau = (\tau^* - \tau_0)/\tau_0$, $\Delta_\tau \in [(T_1-\tau_0)/\tau_0, (T_2-\tau_0)/\tau_0]$.

Let us introduce into consideration quantity $z_r^2 = 2a_0^2 T_2/N_0$ representing SNR for the rectangular pulse with amplitude a_0 and duration T_2. Let us express SNR z_0^2 for the received signal in terms of SNR z_r^2

$$z_0^2 = z_r^2 T_0 \frac{1+bT_0+b^2T_0^2/3}{1+b+b^2/3}. \tag{16}$$

Let us choose the true value of signal duration in the middle of a priori interval $\tau_0 = (T_1+T_2)/2$, then $T_0 = (k+1)/2k$, where $k = T_2/T_1$ is the dynamic range of variation of the unknown duration.

Figure 2 presents the relationships of quantity χ (13) as a function of Δ_τ. Quantity χ characterizes the loss in accuracy of the QL estimate of amplitude (5) as compared with the accuracy of the ML estimate of amplitude (11) for the rectangular pulse with beveled peak (14) at SNR $z_r = 5$ and different tilts of the beveled peak of the pulse. Solid line corresponds to $b = 0$ (rectangular pulse), dashed line corresponds to $b = 1$, dotted line corresponds to $b = -0.5$. From Fig. 2 it can be seen that in the case of the growing pulse peak ($b = 1$) the loss in accuracy of the QL estimate is higher than in the case of decreasing pulse peak ($b = -0.5$).

Figure 3 displays the relationships of quantity χ (13) as a function of Δ_τ for the rectangular pulse with beveled peak (14) at $b = 1$ and several values of SNR. Solid line depicts the relationship for SNR $z_r = 5$, dotted line for $z_r = 8$, and dashed line for $z_r = 12$. From Fig. 3 it can be seen that at the value of parameter $\Delta_\tau \le 0$ the loss in accuracy of the QL estimate does not depend on SNR. For values $\Delta_\tau > 0$ the loss in accuracy of the QL estimate rapidly increases with the rise of SNR. In accordance with Figs. 2 and 3 the unknown duration of the received signal may lead to a significant deterioration of the accuracy of amplitude estimate.

In order to enhance the accuracy of amplitude estimation, we can apply the ML algorithm based on the search for the position of absolute maximum of LRF logarithm

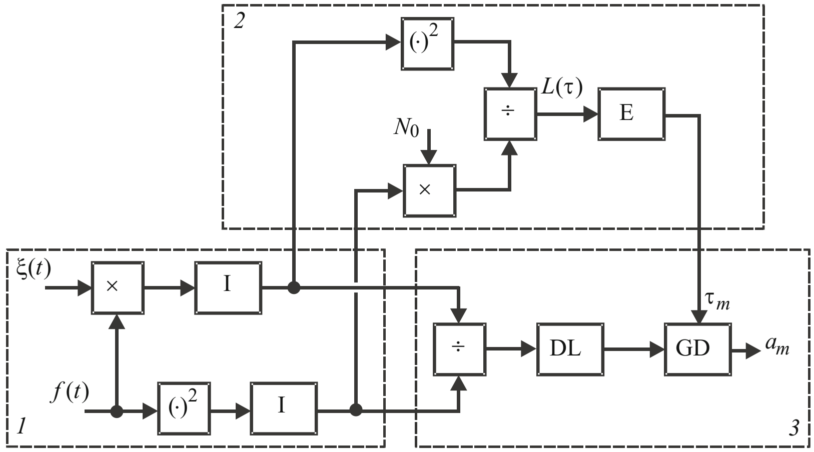

Fig. 4.

$$a_m = \arg\sup L(a), \tag{17}$$

where $L(a) = L(a, \tau_m) = \sup_\tau L(a, \tau)$ is the LRF logarithm, where the unknown duration is replaced with its maximum likelihood estimate τ_m that is equivalent to maximization of expression (2) in terms of duration. In turn, the ML estimate of duration is determined as the position of maximum of LRF logarithm [6]

$$\tau_m = \arg\sup L(\tau), \tag{18}$$

where $L(\tau) = L(\hat{a}, \tau) = \sup_a L(a, \tau)$. As can be seen from equation $\left.\frac{\partial L(a,\tau)}{\partial a}\right|_{\hat{a}} = 0$, similar to expression (5), LRF logarithm (2) reaches its maximum value when

$$\hat{a}(\tau) = \int_0^\tau \xi(t) f(t) \mathrm{d}t \Big/ \int_0^\tau f^2(t) \mathrm{d}t. \tag{19}$$

Substituting expression (19) into (2) we find

$$L(\tau) = \sup_a L(a, \tau) = \left(\int_0^\tau \xi(t) f(t) \mathrm{d}t \right)^2 \Big/ N_0 \int_0^\tau f^2(t) \mathrm{d}t. \tag{20}$$

Taking into account expressions (18) and (19), for the ML amplitude estimate (17) of signal with unknown duration we have:

$$a_m = \hat{a}(\tau_m) = \int_0^{\tau_m} \xi(t) f(t) \mathrm{d}t \Big/ \int_0^{\tau_m} f^2(t) \mathrm{d}t, \tag{21}$$

where τ_m is the ML estimate of duration of signal with unknown amplitude.

Figure 4 presents a block diagram of the device generating the ML estimate of amplitude of the received signal. Blocks in frames *1* and *2* (Fig. 4) generate the duration estimate $\tau_m = \arg\sup L(\tau)$ on the basis of expressions (18) and (20) [6], while blocks in frames *1* and *3* produce the amplitude estimate a_m in accordance with expression (21). The following designations are used in this diagram: I are the integrators performing integration on time interval [0, t], where t takes on values from interval [0; T_2], E is the

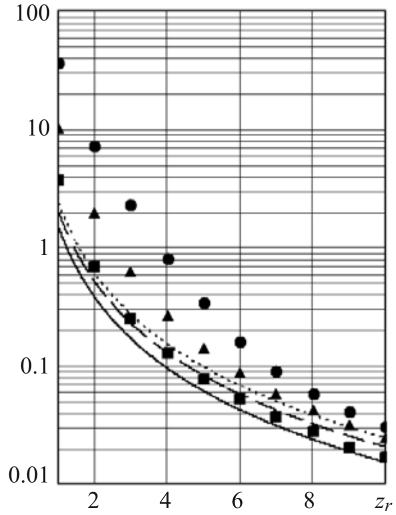

Fig. 5.

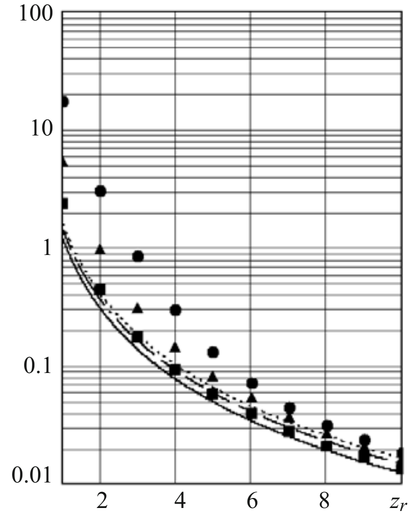

Fig. 6.

extremator that determines τ_m, i.e., the position of maximum value of the input signal; DL is the delay line performing the signal delay by time T_2, GD is the gating device performing the sampling of a generated value of amplitude estimate a_m at the instant of time τ_m.

Signal (1) is a regular (differentiable) function of amplitude a and discontinuous function of duration τ. Thus, the regularity conditions for signal (1) are partially breached [2]. As shown in paper [7], the accuracy of the ML estimate of regular parameter (amplitude) asymptotically (with the rise of SNR) does not depend on a priori lack of knowledge of the discontinuous parameter (duration). This means that the bias and scattering of ML amplitude estimate (17) at large values of SNR asymptotically coincide with the bias and scattering (12) of amplitude estimation (11) of signal with a priori known duration. That is why the relationships presented in Figs. 2 and 3 can be interpreted as quantities characterizing the gain in accuracy of ML estimate (17) as compared with the accuracy of QL estimate (4).

In order to verify the performance of the synthesized estimation algorithms and also to establish the application boundaries of obtained asymptotic expressions for the scattering and bias of estimate, the statistical computer simulation of the ML algorithm of amplitude estimate for a rectangular pulse with beveled peak (14) was performed. For simulation the LRF logarithm (20) was presented in the form:

$$L(\eta) = \frac{\left[z_r S(\eta, \eta_0) + N(\eta)\right]^2}{2(1 + b\eta + b^2\eta^2 / 3)},$$

$$S(\eta, \eta_0) = \min(\eta, \eta_0)\, \frac{1 + b\min(\eta, \eta_0) + b^2 \min^2(\eta, \eta_0) / 3}{\sqrt{1 + b + b^2 / 3}},$$

$$N(\eta) = \sqrt{\frac{T_2}{N_0}} \int_0^{\eta} n(T_2 x)(1 + bx)\mathrm{d}x,$$

where $\eta = \tau / T_2$, $\eta_0 = \tau_0 / T_2$. The simulation with step $\Delta\eta = 10^{-6}$ involved the generation of function $N(\eta)$ samples, on the basis of which the realization of LRF logarithm was approximated by the step function with maximum relative root mean square error $\varepsilon = 0.1$. Discrete samples of the LRF logarithm were presented in the form:

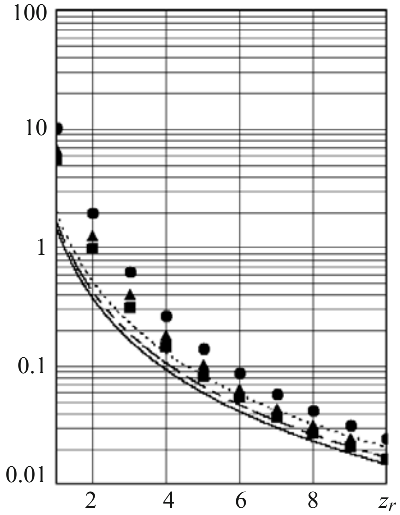

Fig. 7.

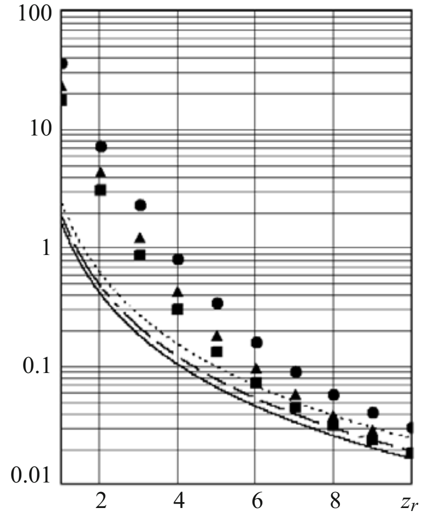

Fig. 8.

$$L(n\Delta\eta) = \frac{\left[z_r S(n\Delta\eta, n_0\Delta\eta) + \sqrt{\Delta\eta/2} \sum_{k=1}^{n} (1 + bk\Delta\eta) x_k \right]^2}{2(1 + bn\Delta\eta + b^2 n^2 \Delta\eta^2 / 3)},$$

where x_k are the Gaussian statistically independent random quantities with zero mathematical expectation and unit dispersion, $n = n_1, n_2$, $n_1 = 1/k\Delta\eta$, $n_2 = 1/\Delta\eta$, $n_0 = \eta_0/\Delta\eta$. The following quantities were generated in the ith test on the basis of samples $L_i(n\Delta\eta)$:

$$n_{mi} = \operatorname{argsup} L_i(n\Delta\eta), \quad \eta_{mi} = n_{mi}\Delta\eta.$$

Next, the normalized ML amplitude estimate was generated:

$$\frac{a_{mi}}{a_0} = \frac{\left[S(n_{mi}\Delta\eta, n_0\Delta\eta) + \frac{\sqrt{\Delta\eta}}{z_r} \sum_{k=1}^{n_{mi}} (1 + bk\Delta\eta) x_k \right]}{2(1 + bn_{mi}\Delta\eta + b^2 n_{mi}^2 \Delta\eta^2 / 3)}.$$

The simulation process involved the realization of $N = 10^5$ test cycles. The experimental values of the normalized bias and scattering of the ML duration estimate were calculated by formulas:

$$\frac{B_a}{a_0} = \frac{1}{N} \sum_{k=1}^{N} \left(\frac{a_{mk}}{a_0} - 1 \right),$$

$$\frac{V_a}{a_0^2} = \frac{1}{N} \sum_{k=1}^{N} \left(\frac{a_{mk}}{a_0} - 1 \right)^2.$$

Figures 5–8 present the scatterings of amplitude estimates normalized in terms of quantity a_0^2. Markers designate the experimental normalized scatterings of the ML amplitude estimate of signal with unknown duration (17). Solid, dashed and dotted lines designate the theoretical values of normalized scatterings of the ML estimates that are calculated by formulas (12).

The relationships shown in Figs. 5 and 6 were built for the increasing pulse with tilt $b = 0.6$ and the decreasing pulse with tilt $b = -0.3$, respectively. Solid curves in Figs. 5 and 6 were built for $k = 2$, dashed curves for $k = 5$, and dotted ones for $k = 20$. Squares, triangles, and circles plotted the relationships of the normalized scatterings of the ML estimate of signal amplitude that were obtained during the simulation for $k = 2$, 5, and 20, respectively.

Figures 7 and 8 display the relationships of the normalized scatterings of amplitude estimate at $k = 5$ and $k = 20$, respectively. Solid lines in Figs. 7 and 8 were built for $b = -0.3$, dashed lines for $b = 0$, and dotted lines for $b = 0.6$. Squares, triangles, and circles plotted the relationships of the normalized scatterings of the ML estimate of signal amplitude that were obtained during the simulation for $b = -0.3$, 0, and 0.6, respectively.

As could be expected, with the rise of k characterizing the dynamic range of variation of unknown duration the scattering of the ML amplitude estimate increases. It is clear that with the rise of SNR the scattering of the ML amplitude estimate in case of the unknown duration asymptotically converges to the scattering of the ML amplitude estimate in case of the known duration. Asymptotic values of the amplitude estimate scattering are in better agreement with the simulation results for the signal with a decreasing peak.

The obtained results of the synthesis and analysis of the algorithms of amplitude estimation of signal with unknown duration allow us to make a sound choice of the necessary estimation algorithm depending on the available a priori information about the signal duration and also on requirements regarding the simplicity of the algorithm realization and requirements regarding the estimation accuracy.

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