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ESTIMATION OF THE NUMBER OF RADIO SIGNALS WITH UNKNOWN AMPLITUDES AND PHASES

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Some algorithms for estimating the number of radio signals with known and unknown amplitudes and phases are synthesized and analyzed. Modified maximum-likelihood algorithms are used to obtain consistent algorithms for estimating the number of radio signals in the case of unknown amplitudes and phases. Efficiency of the estimation algorithms is quantitatively characterized by the abridged probability of the signal-number estimation error. The studied-algorithm parameters are optimized according to the analysis results.

1. INTRODUCTION

The necessity of estimating the number of received signals emerges when solving various problems of statistical radiophysics and radioengineering. For example, when using a multipath radio channel, e.g., in MIMO systems [1, 2], the number of radio paths is *a priori* unknown and should be determined. During the radar and acoustic-location (active or passive) observation, the case where the number of sources of the signals arriving at the antenna array is unknown is rather widespread [3–9]. However, the problem of estimating the number of signals has not yet been completely solved. There appear difficulties in determining the estimation-algorithm structure and the results of theoretical analysis of the operation quality of the algorithms for estimating the number of signals are in fact absent. Moreover, the commonly accepted and correct quantitative characterization of such algorithms has not been developed. Lack of the quantitative characteristics of the algorithms for estimating the number of signals makes it difficult to compare the algorithms and choose the most efficient one.

Some algorithms for estimating the number of radio signals with known and unknown amplitudes and phases are considered below. The probability of the signal-number estimation is used as the algorithm-efficiency characteristic.

2. RADIO SIGNALS WITH KNOWN AMPLITUDES AND PHASES

Let us assume that we observe the sum of ν narrowband radio signals $s_i(t, a_i, \varphi_i) = a_i f_i(t) \cos(\omega_i t + \Psi_i(t) - \varphi_i)$ and, as a result, the following signal is received:

$$s(t, \nu, \mathbf{a}_\nu, \boldsymbol{\varphi}_\nu) = \sum_{i=1}^{\nu} s_i(t, a_i, \varphi_i) = \sum_{i=1}^{\nu} a_i f_i(t) \cos(\omega_i t + \Psi_i(t) - \varphi_i), \tag{1}$$

where $\nu = 1, \ldots, \nu_{\max}$, a_i and ω_i are the amplitude and frequency of the ith signal, respectively (a_i and ω_i are the real numbers), $\varphi_i \in [0, 2\pi]$ is the signal phase, $\Psi_i(t)$ is the signal phase-modulation law, $f_i(t)$ is the signal envelope, and $\mathbf{a}_\nu = (a_i, \ldots, a_\nu)$ and $\boldsymbol{\varphi}_\nu = (\varphi_i, \ldots, \varphi_\nu)$ are the notations used in what follows.

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Let signal (1) be observed during the time interval $[T_1, T_2]$ against the background of additive Gaussian white noise $n(t)$ with one-sided spectral density N_0. Therefore, processing can be performed for the realization

$$x(t) = n(t) + \sum_{i=1}^{\nu_0} a_{0i} f_i(t) \cos(\omega_i t + \Psi_i(t) - \varphi_{0i}), \tag{2}$$

where ν_0 is the true number of signals in Eq. (1) and the sets $\{a_{0i}\}_{i=1}^{\nu_{\max}} = (a_{01}, \ldots, a_{0\nu_{\max}})$ and $\{\varphi_{0i}\}_{i=1}^{\nu_{\max}} = (\varphi_{01}, \ldots, \varphi_{0\nu_{\max}})$ contain the true values of the signal amplitudes and phases.

The maximum-likelihood method is used for estimating the number ν_0 of signals. In [10], the following formula for the logarithm of the likelihood-ratio functional (LRF) is proposed if the interference is additive white Gaussian noise:

$$L(l) = \frac{2}{N_0} \int_{T_1}^{T_2} x(t) s(t, l)\, \mathrm{d}t - \frac{1}{N_0} \int_{T_1}^{T_2} s^2(t, l)\, \mathrm{d}t. \tag{3}$$

Here, l denotes a family of unknown parameters of the signal $s(t, l)$.

Substituting Eq. (1) into Eq. (3), we write the LRF logarithm

$$L(\nu, \mathbf{a}_\nu, \boldsymbol{\varphi}_\nu) = \frac{2}{N_0} \sum_{m=1}^{\nu} a_m \int_{T_1}^{T_2} x(t) f_m(t) \cos(\omega_m t + \Psi_m(t) - \varphi_m)\, \mathrm{d}t - \frac{1}{N_0} \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} a_i a_j K_{ij}. \tag{4}$$

Here,$\nu = 1, \ldots, \nu_{\max}$ and

$$K_{ij} = \frac{1}{2} \int_{T_1}^{T_2} f_i(t) f_j(t) \cos[\omega_i t + \Psi_i(t) - \varphi_i - \omega_j t - \Psi_j(t) + \varphi_j]\, \mathrm{d}t$$
$$+ \frac{1}{2} \int_{T_1}^{T_2} f_i(t) f_j(t) \cos[\omega_i t + \Psi_i(t) - \varphi_i + \omega_j t + \Psi_j(t) - \varphi_j]\, \mathrm{d}t \tag{5}$$

is the scalar product of the functions $f_i(t) \cos[\omega_i t + \Psi_i(t) - \varphi_i]$ and $f_j(t) \cos[\omega_j t + \Psi_j(t) - \varphi_j]$.

Let us consider the case where the signals in sum (1) satisfy the narrowbandness condition [10, 11]

$$\omega_i\, (T_2 - T_1) \gg 1, \qquad i = 1, \ldots, \nu. \tag{6}$$

In this case, the second term in Eq. (5) is small compared with the first term for all i and j, which allows us to rewrite Eq. (5) in the form

$$K_{ij} = V_{cij} \cos(\varphi_i - \varphi_j) + V_{sij} \sin(\varphi_i - \varphi_j), \tag{7}$$

where

$$V_{cij} = \frac{1}{2} \int_{T_1}^{T_2} f_i(t) f_j(t) \cos[(\omega_i - \omega_j)\, t + \Psi_i(t) - \Psi_j(t)]\, \mathrm{d}t,$$

$$V_{sij} = \frac{1}{2} \int_{T_1}^{T_2} f_i(t) f_j(t) \sin[(\omega_i - \omega_j)\, t + \Psi_i(t) - \Psi_j(t)]\, \mathrm{d}t.$$

Using Eq. (4), we write the maximum-likelihood algorithm for estimating the number of signals as

$$\hat{\nu}\colon\ L_0(\hat{\nu}) = \sup_{\nu} L_0(\nu), \qquad \nu = 1, \ldots, \nu_{\max}, \tag{8}$$

where $L_0(\nu) = L(\nu, \mathbf{a}_{0\nu}, \boldsymbol{\varphi}_{0\nu})$.

Let us consider the properties of LRF logarithm (4). For this purpose, we substitute realization (2) of the observed data into Eq. (4) and obtain

$$L_0(\nu) = \sum_{j=1}^{\nu} z_j \sum_{i=1}^{\nu_0} z_i \rho_{ij} + \sum_{j=1}^{\nu} z_j \xi_j - \frac{1}{2} \sum_{j=1}^{\nu} \sum_{i=1}^{\nu} z_i z_j \rho_{ij}. \tag{9}$$

Here,

$$\xi_i = \sqrt{\frac{2}{N_0 K_{ii}}} \int_{T_1}^{T_2} n(t) f_i(t) \cos(\omega_i t + \Psi_i(t) - \varphi_{0i})\, \mathrm{d}t, \qquad \rho_{ij} = \frac{K_{ij}}{\sqrt{K_{ii} K_{jj}}}, \qquad \boldsymbol{\rho}_\nu = \|\rho_{ij}\|_{i=1}^{\nu}{}_{j=1}^{\nu} and$$

$z_i^2 = 2a_{0i}^2 K_{ii}/N_0$ is the signal-to-noise ratio (SNR) for the ith signal. With allowance for Eq. (7), the coefficient of correlation between ith and the jth signals can be represented as

$$\rho_{ij} = \rho_{\mathrm{c}ij} \cos(\varphi_i - \varphi_j) + \rho_{\mathrm{s}ij} \sin(\varphi_i - \varphi_j), \tag{10}$$

where

$$\rho_{\mathrm{c}ij} = 2V_{\mathrm{c}ij} \Bigg/ \sqrt{\int_{T_1}^{T_2} f_i^2(t)\, \mathrm{d}t \int_{T_1}^{T_2} f_j^2(t)\, \mathrm{d}t}, \qquad \rho_{\mathrm{s}ij} = 2V_{\mathrm{s}ij} \Bigg/ \sqrt{\int_{T_1}^{T_2} f_i^2(t)\, \mathrm{d}t \int_{T_1}^{T_2} f_j^2(t)\, \mathrm{d}t}.$$

Using Eqs. (9) and (10), we obtain the auxiliary relationships

$$\begin{aligned} L_0(\nu_0) - L_0(\nu_0 + m) &= -\sum_{i=\nu_0+1}^{\nu_0+m} z_i \xi_i + \frac{1}{2} \sum_{i=\nu_0+1}^{\nu_0+m} \sum_{j=\nu_0+1}^{\nu_0+m} z_i z_j \rho_{ij}, \\ L_0(\nu_0) - L_0(\nu_0 - m) &= \sum_{i=\nu_0-m+1}^{\nu_0} z_i \xi_i + \frac{1}{2} \sum_{i=\nu_0-m+1}^{\nu_0} \sum_{j=\nu_0-m+1}^{\nu_0} z_i z_j \rho_{ij}. \end{aligned} \tag{11}$$

Logarithm (9) is written as the sum of signal and noise components:

$$L_0(\nu) = S(\nu) + N(\nu),$$

where

$$S(\nu) = \langle L_0(\nu) \rangle = \sum_{j=1}^{\nu} z_j \sum_{i=1}^{\nu_0} z_i \rho_{ij} - \frac{1}{2} \sum_{j=1}^{\nu} \sum_{i=1}^{\nu} z_i z_j \rho_{ij}, \qquad N(\nu) = L_0(\nu) - \langle L_0(\nu) \rangle.$$

From the above expression and Eq. (11), we have

$$S(\nu_0) - S(\nu_0 + m) = \frac{1}{2} \sum_{i=\nu_0+1}^{\nu_0+m} \sum_{j=\nu_0+1}^{\nu_0+m} z_i z_j \rho_{ij}, \qquad S(\nu_0) - S(\nu_0 - m) = \frac{1}{2} \sum_{i=\nu_0-m+1}^{\nu_0} \sum_{j=\nu_0-m+1}^{\nu_0} z_i z_j \rho_{ij}.$$

Taking into account these relationships and the fact that the matrix $\boldsymbol{\rho}_\nu$ is nonnegative-definite, we can conclude that the quantity ν_0 corresponds to the upper limit $S(\nu_0) = \sup_\nu S(\nu)$. Therefore, estimate (8) is consistent according to [10].

Efficiency of the algorithm for estimating the number of signals can be characterized by the error probability $p_{\mathrm{e}} = p(\hat{\nu} \neq \nu_0)$. However, the calculation of this probability requires substantial computational resources. To obtain a simplified approximate formula for the error probability, we note that any algorithm $\Re$ for estimating the signal number can be represented as

$$\hat{\nu}\colon\ R[\hat{\nu}; x(t)] = \sup_{\nu} R[\nu; x(t)],$$

where $R[\nu; x(t)]$ is the functional determined by the structure of the algorithm $\Re$ and depending on the number of signals and the realization of the observed data. Using the above expression, the total error probability for the algorithm $\Re$ can be written as

$$p_{\mathrm{e}} = 1 - p\{R[\nu_0; x(t)] > R[i; x(t)],\ i \neq \nu_0,\ i = 1, \ldots, \nu_{\max}\}. \tag{12}$$

Then, as an approximation to the total error probability, we consider the abridged error probability for the algorithm $\Re$, which is determined as

$$p_{\mathrm{t}} = 1 - p\{R[\nu_0; x(t)] > R[\nu_0 + 1; x(t)], R[\nu_0; x(t)] > R[\nu_0 - 1; x(t)]\}. \tag{13}$$

Equation (13) shows that the abridged error probability is the lower boundary for the total error probability (12) if $1 < \nu_0 < \nu_{\max}$. It should also be noted that the abridged error probability coincides with the total error probability if $\nu_{\max} = 3$ and $\nu_0 = 2$.

Let us find the abridged error probability (13) for algorithm (8):

$$p_{\mathrm{t}0} = 1 - p[L_0(\nu_0) > L_0(\nu_0 + 1), L_0(\nu_0) > L_0(\nu_0 - 1)]. \tag{14}$$

Using Eq. (11) for $m = 1$, we rewrite Eq. (14) in the form

$$p_{\mathrm{t}0} = 1 - p(\xi_{\nu_0} > -z_{\nu_0}/2, \xi_{\nu_0+1} < z_{\nu_0+1}/2). \tag{15}$$

Taking into account that ξ_{ν_0} and ξ_{ν_0+1} are the Gaussian random quantities with zero mathematical expectations, unit variances, and the correlation coefficient $\langle \xi_{\nu_0}\xi_{\nu_0+1}\rangle = \rho_{\nu_0,\nu_0+1}$, we obtain the following formula for calculating the abridged error probability (15) for algorithm (8):

$$p_{\mathrm{t}0} = 1 - \frac{1}{2\pi\sqrt{1 - \rho^2_{\nu_0,\nu_0+1}}} \int\limits_{-\infty}^{z_{\nu_0+1}/2} \int\limits_{-z_{\nu_0}/2}^{\infty} \exp\left[-\frac{x^2 - 2xy\rho_{\nu_0,\nu_0+1} + y^2}{2\,(1 - \rho^2_{\nu_0,\nu_0+1})}\right] \mathrm{d}x\, \mathrm{d}y. \tag{16}$$

After the change of variables, Eq. (16) is rewritten as

$$p_{\mathrm{t}0} = 1 - \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{z_{\nu_0+1}/2} \exp\left(-\frac{y^2}{2}\right) \tilde{\Phi}\left(\frac{z_{\nu_0}/2 + \rho_{\nu_0,\nu_0+1}y}{\sqrt{1 - \rho^2_{\nu_0,\nu_0+1}}}\right) \mathrm{d}y, \tag{17}$$

where $\tilde{\Phi}(x) = \int_{-\infty}^{x} \exp(-t^2/2)\,\mathrm{d}t/\sqrt{2\pi}$ is the probability integral.

Let us consider the special case where the signals in Eq. (1) are orthogonal. In this case, $\rho_{\nu_0,\nu_0+1} = 0$ and the abridged error probability (16) takes the form

$$p_{\mathrm{t}0} = 1 - \tilde{\Phi}(z_{\nu_0}/2)\tilde{\Phi}(z_{\nu_0+1}/2). \tag{18}$$

If $z^2_{\nu_0+1} = z^2_{\nu_0} = z^2$, then for a sufficiently large value of z^2, instead of Eq. (18), one can use the asymptotic formula

$$p_{\mathrm{t}0} \approx \frac{2}{z}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{z^2}{8}\right). \tag{19}$$

We now consider the case where $\rho_{\nu_0,\nu_0+1} \to 1$, i.e., the signals $s_{\nu_0}(t)$ and $s_{\nu_0+1}(t)$ coincide. Then Eq. (16) is written as

$$p_{t0} = 1 - \frac{1}{\sqrt{2\pi}} \int\limits_{-z_{\nu_0}/2}^{z_{\nu_0+1}/2} \exp\left(-\frac{y^2}{2}\right) dy$$
$$= 2 - \tilde{\Phi}\left(\frac{z_{\nu_0+1}}{2}\right) - \tilde{\Phi}\left(\frac{z_{\nu_0}}{2}\right). \quad (20)$$

If we put $z_{\nu_0+1}^2 = z_{\nu_0}^2 = z^2$ and assume that the value of z^2 is sufficiently large, then asymptotic formula (19) is obtained for the probability given by in Eq. (20). Therefore, for a sufficiently large signal-to-noise ratio z^2, the abridged error probabilities in the cases $\rho_{\nu_0,\nu_0+1} \to 0$ and $\rho_{\nu_0,\nu_0+1} \to 1$ coincide. The latter result is attributed to the fact that the cases with different numbers of signals (even identically shaped) are significantly different from the energy viewpoint when the signal amplitudes and phases are known and the SNR is sufficiently large.

Consider the special case where all the functions of the set $\{f_i(t)\}_{i=1}^{\nu_{\max}}$ are identically equal to unity in the interval $T_1 \le t \le T_2$ and $\Psi_i(t) = 0$. Then the radio signals in Eq. (1) are the segments of the harmonic oscillations. We also assume that for any $1 \le k \le \nu_{\max}$, the equality $\omega_k = k\omega$, where ω is the real number, is fulfilled. If the signals in Eq. (1) satisfy the narrowbandness condition specified by Eq. (6), then Eq. (10) for the correlation coefficient of the ith and jth signals ($i \ne j$) is written in the form

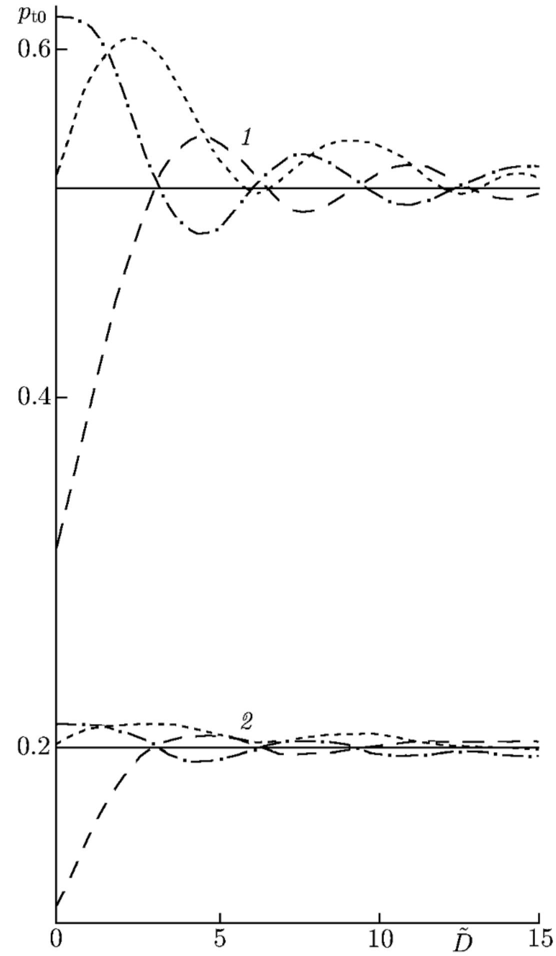

Fig. 1. Error probability for known amplitudes and phases.

$$\rho_{ij} = \frac{\sin[2\pi\,(i-j)\,\tilde{D}]}{2\pi\,(i-j)\,\tilde{D}} \cos(\varphi_i - \varphi_j) + \frac{\sin^2[2\pi\,(i-j)\,\tilde{D}]}{\pi\,(i-j)\,\tilde{D}} \sin(\varphi_i - \varphi_j),$$

where $\tilde{D} = \omega T/(2\pi)$. Figure 1 shows theoretical dependences of the abridged error probability (16) on $\tilde{D}$ in the cases of in-phase signals ($\varphi_{i+1} - \varphi_i = 0$; dash–dot lines), quadrature signals ($\varphi_{i+1} - \varphi_i = \pi/2$; dotted lines), and the antiphase signals ($\varphi_{i+1} - \varphi_i = \pi$; dashed lines). In addition, the solid lines in Fig. 1 show the abridged error probability (18) for signal-number estimation algorithm (8) in the case where the signals are orthogonal . The line sets *1* and *2* correspond to the SNRs $z_{\nu_0} = z_{\nu_0+1} = 1$ and $z_{\nu_0} = z_{\nu_0+1} = 2.5$, respectively. Figure 1 shows that when estimating the number of signals the abridged error probability (17) tends to the abridged error probability (18) for the orthogonal signals with increasing $\tilde{D}$ in all the three considered cases. The difference between the initial radio-signal phases does not significantly influence the error-probability value for $\tilde{D} > 4$.

3. PROPERTIES OF THE LRF LOGARITHM

Let us consider the case where the received-signal amplitudes and phases are *a priori* unknown. The following notations are used:

$$\begin{pmatrix} G_{cij} \\ G_{sij} \end{pmatrix} = \int\limits_{T_1}^{T_2} f_i(t) f_j(t) \begin{pmatrix} \cos[\omega_i t + \Psi_i(t)] & \cos[\omega_j t + \Psi_j(t)] \\ \sin[\omega_i t + \Psi_i(t)] & \sin[\omega_j t + \Psi_j(t)] \end{pmatrix} dt,$$

$$\begin{pmatrix} G_{\mathrm{sc}ij} \\ G_{\mathrm{cs}ij} \end{pmatrix} = \int\limits_{T_1}^{T_2} f_i(t) f_j(t) \begin{pmatrix} \sin[\omega_i t + \Psi_i(t)] & \cos[\omega_j t + \Psi_j(t)] \\ \cos[\omega_i t + \Psi_i(t)] & \sin[\omega_j t + \Psi_j(t)] \end{pmatrix} \mathrm{d}t, \tag{21}$$

$$X_{\mathrm{c}i} = \int\limits_{T_1}^{T_2} x(t) f_i(t) \cos[\omega_i t + \Psi_i(t)]\, \mathrm{d}t, \qquad X_{\mathrm{s}i} = \int\limits_{T_1}^{T_2} x(t) f_i(t) \sin[\omega_i t + \Psi_i(t)]\, \mathrm{d}t,$$

$$A_{\mathrm{c}i} = a_i \cos\varphi_i, \qquad A_{\mathrm{s}i} = a_i \sin\varphi_i. \tag{22}$$

Here, $A_{\mathrm{c}i}$ and $A_{\mathrm{s}i}$ are the signal quadratures: $i = 1, \ldots, \nu$ and $j = 1, \ldots, \nu$. Quantities (21) and (22) are the elements of the matrices $\mathbf{G}_{\mathrm{c}\nu}$, $\mathbf{G}_{\mathrm{s}\nu}$, $\mathbf{G}_{\mathrm{cs}\nu}$, and $\mathbf{G}_{\mathrm{sc}\nu}$ and the vectors $\mathbf{X}_{\mathrm{c}\nu}$, $\mathbf{X}_{\mathrm{s}\nu}$, $\mathbf{A}_{\mathrm{c}\nu}$, and $\mathbf{A}_{\mathrm{s}\nu}$, respectively. In the case of narrowband radio signals (6), the following relationships are valid for the elements (21):

$$G_{\mathrm{c}ij} = G_{\mathrm{s}ij} = V_{\mathrm{c}ij}, \qquad G_{\mathrm{sc}ij} = G_{\mathrm{cs}ij} = V_{\mathrm{s}ij}. \tag{23}$$

Let us rewrite the LRF logarithm in Eq. (4) using the above-introduced notations:

$$L(\nu, \mathbf{A}_{\mathrm{c}\nu}, \mathbf{A}_{\mathrm{s}\nu}) = \frac{2}{N_0}(\mathbf{A}_{\mathrm{c}\nu}^{+}\mathbf{X}_{\mathrm{c}\nu} + \mathbf{A}_{\mathrm{s}\nu}^{+}\mathbf{X}_{\mathrm{s}\nu}) - \frac{1}{N_0}(\mathbf{A}_{\mathrm{c}\nu}^{+}\mathbf{G}_{\mathrm{c}\nu}\mathbf{A}_{\mathrm{c}\nu} + \mathbf{A}_{\mathrm{s}\nu}^{+}\mathbf{G}_{\mathrm{s}\nu}\mathbf{A}_{\mathrm{s}\nu} + \mathbf{A}_{\mathrm{c}\nu}^{+}\mathbf{G}_{\mathrm{cs}\nu}\mathbf{A}_{\mathrm{s}\nu} + \mathbf{A}_{\mathrm{s}\nu}^{+}\mathbf{G}_{\mathrm{sc}\nu}\mathbf{A}_{\mathrm{c}\nu}), \tag{24}$$

where the superscript + denotes transposition. Denote

$$\mathbf{A}_{\mathrm{q}\nu} = \begin{pmatrix} \mathbf{A}_{\mathrm{c}\nu} \\ \mathbf{A}_{\mathrm{s}\nu} \end{pmatrix}, \qquad \mathbf{X}_{\nu} = \begin{pmatrix} \mathbf{X}_{\mathrm{c}\nu} \\ \mathbf{X}_{\mathrm{s}\nu} \end{pmatrix}, \qquad \mathbf{G}_{\nu} = \begin{pmatrix} \mathbf{G}_{\mathrm{c}\nu} & \mathbf{G}_{\mathrm{cs}\nu} \\ \mathbf{G}_{\mathrm{sc}\nu} & \mathbf{G}_{\mathrm{s}\nu} \end{pmatrix},$$

so that the LRF logarithm in Eq. (24) takes the form

$$L(\nu, \mathbf{A}_{\mathrm{q}\nu}) = \frac{2}{N_0}\mathbf{A}_{\mathrm{q}\nu}^{+}\mathbf{X}_{\nu} - \frac{1}{N_0}\mathbf{A}_{\mathrm{q}\nu}^{+}\mathbf{G}_{\nu}\mathbf{A}_{\mathrm{q}\nu}. \tag{25}$$

Using Eq. (25), we find the maximum-likely estimate

$$\hat{\mathbf{A}}_{\mathrm{q}\nu} = \mathbf{G}_{\nu}^{-1}\mathbf{X}_{\nu}$$

of the signal-quadrature vector for an arbitrary value of the parameter ν.

Substituting the vector $\hat{\mathbf{A}}_{\mathrm{q}\nu}$ into Eq. (25), we obtain an expression for the LRF logarithm maximized by unknown signal quadratures:

$$L_{\mathrm{m}}(\nu) = \frac{1}{N_0}\mathbf{X}_{\nu}^{+}\mathbf{G}_{\nu}^{-1}\mathbf{X}_{\nu} = \frac{1}{N_0}\begin{pmatrix} \mathbf{X}_{\mathrm{c}\nu} \\ \mathbf{X}_{\mathrm{s}\nu} \end{pmatrix}^{+}\begin{pmatrix} \mathbf{G}_{\mathrm{c}\nu} & \mathbf{G}_{\mathrm{cs}\nu} \\ \mathbf{G}_{\mathrm{sc}\nu} & \mathbf{G}_{\mathrm{s}\nu} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{X}_{\mathrm{c}\nu} \\ \mathbf{X}_{\mathrm{s}\nu} \end{pmatrix}. \tag{26}$$

With allowance for Eq. (23), Eq. (26) for the narrowband radio signals can be written as

$$L_{\mathrm{m}}(\nu) = \frac{1}{N_0}\begin{pmatrix} \mathbf{X}_{\mathrm{c}\nu} \\ \mathbf{X}_{\mathrm{s}\nu} \end{pmatrix}^{+}\begin{pmatrix} \mathbf{V}_{\mathrm{c}\nu} & \mathbf{V}_{\mathrm{s}\nu}^{+} \\ \mathbf{V}_{\mathrm{s}\nu} & \mathbf{V}_{\mathrm{c}\nu} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{X}_{\mathrm{c}\nu} \\ \mathbf{X}_{\mathrm{s}\nu} \end{pmatrix},$$

where $\mathbf{V}_{\mathrm{c}\nu}$ and $\mathbf{V}_{\mathrm{s}\nu}$ are the matrices of the elements $V_{\mathrm{c}ij}$ and $V_{\mathrm{s}ij}$, respectively.

As a result, the algorithm of the maximum-likely estimate of the number of radio signals with the unknown amplitudes and phases takes the form

$$\hat{\nu}\colon L_{\mathrm{m}}(\hat{\nu}) = \sup_{\nu} L_{\mathrm{m}}(\nu), \qquad \nu = 1, \ldots, \nu_{\max}. \tag{27}$$

In what follows, it is shown (see Eq. (44)) that function (26) does not decrease with increasing number ν of signals, so that direct use of the maximum-likelihood method for estimating the number of radio signals with unknown amplitudes and phases is impossible. Therefore, instead of the maximum-likelihood algorithm (27), its modifications are used. The following algorithms can be referred to such modifications.

1. The algorithm with a linear penalty function:

$$L_{\rm D}(\nu;\mathbf{X}_\nu,\kappa)=L_{\rm m}(\nu)-\kappa\nu,\qquad \kappa>0;\qquad \hat\nu\colon\ L_{\rm D}(\hat\nu;\mathbf{X}_\nu,\kappa)=\sup_\nu L_{\rm D}(\nu;\mathbf{X}_\nu,\kappa),\qquad \nu=1,\,\ldots\,,\nu_{\max}. \tag{28}$$

The algorithm for estimating the number of signals by the AIC criterion [7] is the special case of algorithm (28) for $\kappa=2$.

2. The algorithm with a random penalty function [2]:

$$L_{\rm D1}(\nu;\mathbf{X}_\nu,\kappa_1)=L_{\rm m}(\nu)-\kappa_1\nu\,\max_i\left[\frac{1}{N_0}\left(\frac{X_{{\rm c}i}^2}{G_{{\rm c}ii}}+\frac{X_{{\rm s}i}^2}{G_{{\rm s}ii}}\right)\right],\qquad \kappa_1>0,\qquad 1\le i\le\nu_{\max};$$
$$\hat\nu\colon\ L_{\rm D1}(\hat\nu;\mathbf{X}_\nu,\kappa_1)=\sup_\nu L_{\rm D1}(\nu;\mathbf{X}_\nu,\kappa_1),\qquad \nu=1,\,\ldots\,,\nu_{\max}. \tag{29}$$

3. The algorithm with an invariant random penalty function, which is proposed in this work:

$$L_{\rm D2}(\nu;\mathbf{X}_\nu,\kappa_2)=L_{\rm m}(\nu)-\kappa_2\nu\,\max_i[L_{\rm m}(i)-L_{\rm m}(i-1)],\qquad \kappa_2>0,\qquad 1\le i\le\nu_{\max};$$
$$\hat\nu\colon\ L_{\rm D2}(\hat\nu)=\sup_\nu L_{\rm D2}(\nu),\qquad \nu=1,\,\ldots\,,\nu_{\max}. \tag{30}$$

In the case where the signals are orthogonal, algorithm (30) reduces to algorithm (29) with the random penalty function.

4. The algorithm with a inverse penalty function, which is proposed in this work:

$$L_{\rm B}(\nu;\mathbf{X}_\nu,n)=\frac{L_{\rm m}^n(\nu)}{\nu},\qquad n>1;$$
$$\hat\nu\colon\ L_{\rm B}(\hat\nu;\mathbf{X}_\nu,n)=\sup_\nu L_{\rm B}(\nu;\mathbf{X}_\nu,n),\qquad \nu=1,\,\ldots\,,\nu_{\max}. \tag{31}$$

Note that all the above algorithms depend on certain parameters κ, κ_1, κ_2, and n, which should have particular numerical values to ensure practical use of the algorithms. In what follows, when analyzing algorithms (28)–(31), their abridged error probabilities will be obtained (analytically and by statistical simulation). The study of the abridged error probabilities as functions of the parameters κ, κ_1, κ_2, and n allows one to determine the optimal values of the above parameters from the viewpoint of the minimum abridged error probability. However, it is clear that algorithm (31) is consistent for all values of the parameter $n>1$, whereas algorithms (28), (29), and (30) can become inconsistent for some values of the parameters κ, κ_1, and κ_2, respectively.

To analyze algorithms (28)–(31) in terms of the abridged error probability, the LRF logarithm in Eq. (26) is represented as

$$L_{\rm m}(\nu)=\frac{\mathbf{X}_\nu^+\mathbf{G}_\nu^{-1}\mathbf{X}_\nu}{N_0}=\frac{1}{2}\sum_{i=1}^{\nu}(l_{{\rm c}i}^2+l_{{\rm s}i}^2), \tag{32}$$

where

$$l_{{\rm s}i}^2=\frac{2}{N_0}\left[\begin{pmatrix}\mathbf{X}_{{\rm c}i}\\ \mathbf{X}_{{\rm s}i}\end{pmatrix}^+\begin{pmatrix}\mathbf{G}_{{\rm c}i} & \mathbf{G}_{{\rm cs}i}\\ \mathbf{G}_{{\rm sc}i} & \mathbf{G}_{{\rm s}i}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{X}_{{\rm c}i}\\ \mathbf{X}_{{\rm s}i}\end{pmatrix}-\begin{pmatrix}\mathbf{X}_{{\rm c}i}\\ \mathbf{X}_{{\rm s}(i-1)}\end{pmatrix}^+\begin{pmatrix}\mathbf{G}_{{\rm c}i} & \mathbf{G}_{{\rm cs}i(i-1)}\\ \mathbf{G}_{{\rm sc}(i-1)i} & \mathbf{G}_{{\rm s}(i-1)}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{X}_{{\rm c}i}\\ \mathbf{X}_{{\rm s}(i-1)}\end{pmatrix}\right],$$

$$l_{{\rm c}i}^2=\frac{2}{N_0}\left[\begin{pmatrix}\mathbf{X}_{{\rm c}i}\\ \mathbf{X}_{{\rm s}(i-1)}\end{pmatrix}^+\begin{pmatrix}\mathbf{G}_{{\rm c}i} & \mathbf{G}_{{\rm cs}i(i-1)}\\ \mathbf{G}_{{\rm sc}(i-1)i} & \mathbf{G}_{{\rm s}(i-1)}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{X}_{{\rm c}i}\\ \mathbf{X}_{{\rm s}(i-1)}\end{pmatrix}-\right.$$

$$- \begin{pmatrix} \mathbf{X}_{c(i-1)} \\ \mathbf{X}_{s(i-1)} \end{pmatrix}^{+} \begin{pmatrix} \mathbf{G}_{c(i-1)} & \mathbf{G}_{cs(i-1)} \\ \mathbf{G}_{sc(i-1)} & \mathbf{G}_{s(i-1)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{X}_{c(i-1)} \\ \mathbf{X}_{s(i-1)} \end{pmatrix} \Bigg].$$

Here, the matrices $\mathbf{G}_{csi(i-1)}$ and $\mathbf{G}_{cs(i-1)i}$ have the elements G_{csmk}, where $m = 1, \ldots, i$ and $k = 1, \ldots, i-1$, and the elements G_{csmk}, where $m = 1, \ldots, i-1$, $k = 1, \ldots, i$, respectively, and the matrices $\mathbf{G}_{sci(i-1)}$ and $\mathbf{G}_{sc(i-1)i}$ are specified in a similar way.

To study the properties of the random quantities $\{l_{ci}\}_{i=1}^{\nu_{\max}}$ and $\{l_{si}\}_{i=1}^{\nu_{\max}}$, we formulate and prove the following statement. Let $\{A_i\}_{i=1}^{N}$ be a set of random quantities with finite second moments. We consider the vectors $\mathbf{A}_n$ and the matrices $\mathbf{C}_n$ with the elements A_i and C_{ij}, respectively, where $i = 1, \ldots, n$, $j = 1, \ldots, n$, $C_{ij} = \mathrm{cov}(A_i, A_j) = \langle (A_i - \langle A_i \rangle)(A_j - \langle A_j \rangle) \rangle$ is the correlation moment [11] (or covariance) of the random quantities A_i and A_j, and $n = 1, \ldots, N$. In addition, we define the random function $r(n)$ of the natural variable n and the random quantities $\{B_n\}_{n=1}^{N}$ as follows:

$$r(n) = \mathbf{A}_n^{+} \mathbf{C}_n^{-1} \mathbf{A}_n, \tag{33}$$

$$B_n^2 = \begin{cases} r(1), & n = 1, \\ r(n) - r(n-1), & n \geq 2. \end{cases} \tag{34}$$

Then we formulate Statement 1: the random quantities $\{B_n\}_{n=1}^{N}$, which satisfy Eq. (33), are mutually uncorrelated and have a unit variance.

Let us prove Statement 1. For $n = 1$, the random quantity B_1 is determined by the expression $B_1^2 = A_1^2 / C_{11}$. Let us consider the case $n > 1$ in more detail. Using vector $\mathbf{g}_n$ with the elements C_{in}, where $i = 1, \ldots, n-1$, we rewrite Eq. (33) in the form

$$\begin{aligned} r(n) - r(n-1) &= \mathbf{A}_n^{+} \mathbf{C}_n^{-1} \mathbf{A}_n - \mathbf{A}_{n-1}^{+} \mathbf{C}_{n-1}^{-1} \mathbf{A}_{n-1} \\ &= \left[\begin{pmatrix} \mathbf{A}_{n-1} \\ A_n \end{pmatrix}^{+} \begin{pmatrix} \mathbf{C}_{n-1} & \mathbf{g}_n \\ \mathbf{g}_n^{+} & C_{nn} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{n-1} \\ A_n \end{pmatrix} - \mathbf{A}_{n-1}^{+} \mathbf{C}_{n-1}^{-1} \mathbf{A}_{n-1} \right]. \end{aligned} \tag{35}$$

Using the Frobenius formula [12], we transform the block matrix in Eq. (35) as

$$\begin{pmatrix} \mathbf{C}_{n-1} & \mathbf{g}_n \\ \mathbf{g}_n^{+} & C_{nn} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{C}_{n-1}^{-1} + \mathbf{C}_{n-1}^{-1} \mathbf{g}_n H^{-1} \mathbf{g}_n^{+} \mathbf{C}_{n-1}^{-1} & -\mathbf{C}_{n-1}^{-1} \mathbf{g}_n H^{-1} \\ -H^{-1} \mathbf{g}_n^{+} \mathbf{C}_{n-1}^{-1} & H^{-1} \end{pmatrix}, \tag{36}$$

where $H = C_{nn} - \mathbf{g}_n^{+} \mathbf{C}_{n-1}^{-1} \mathbf{g}_n$ is the Schur complement. It is shown in [13] that if the matrices $\mathbf{C}_n$ and $\mathbf{C}_{n-1}$ are nondegenerate, then $H \neq 0$. Moreover, if the matrix $\mathbf{C}_{n-1}$ is positive-definite, then $H^{-1} > 0$ [12]. Performing identical transformations in Eq. (35) and using Eq. (36), we obtain

$$B_n^2 = r(n) - r(n-1) = (A_n - \mathbf{A}_{n-1}^{+} \mathbf{C}_{n-1}^{-1} \mathbf{g}_n)^2 / H. \tag{37}$$

Then the random quantities $\{B_i\}_{i=1}^{N}$, which satisfy Eq. (37), are written in the form

$$B_n = \gamma_n (A_n - \mathbf{A}_{n-1}^{+} \mathbf{C}_{n-1}^{-1} \mathbf{g}_n)/\sqrt{H}, \quad n > 1; \qquad B_1 = \gamma_1 A_1 / \sqrt{C_{11}}, \tag{38}$$

where the coefficient $\gamma_n = 1$ or $\gamma_n = -1$ for any $1 \leq n \leq N$.

Let $\beta = 1, \ldots, N$, $\alpha = 1, \ldots, N$, and for definiteness we put $\beta > \alpha$. Obviously, $\mathrm{cov}(B_1, B_\beta) = 0$. Let us show that B_α and B_β are also uncorrelated in the case where $\beta > 1$ and $\alpha > 1$:

$$\begin{aligned} \mathrm{cov}(A_\alpha - \mathbf{A}_{\alpha-1}^{+} \mathbf{C}_{\alpha-1}^{-1} \mathbf{g}_\alpha, A_\beta - \mathbf{A}_{\beta-1}^{+} \mathbf{C}_{\beta-1}^{-1} \mathbf{g}_\beta) &= \mathrm{cov}(\mathbf{A}_{\alpha-1}^{+} \mathbf{C}_{\alpha-1}^{-1} \mathbf{g}_\alpha, \mathbf{A}_{\beta-1}^{+} \mathbf{C}_{\beta-1}^{-1} \mathbf{g}_\beta) \\ - \mathrm{cov}(A_\alpha, \mathbf{A}_{\beta-1}^{+} \mathbf{C}_{\beta-1}^{-1} \mathbf{g}_\beta) &- \mathrm{cov}(\mathbf{A}_{\alpha-1}^{+} \mathbf{C}_{\alpha-1}^{-1} \mathbf{g}_\alpha, A_\beta) + \mathrm{cov}(A_\alpha, A_\beta) \end{aligned}$$

$$= \operatorname{cov}(\mathbf{g}^{+}_{\alpha(\beta-1)}\mathbf{C}^{-1}_{(\alpha-1)\,(\beta-1)}\mathbf{A}_{\beta-1}, \mathbf{g}^{+}_{\beta}\mathbf{C}^{-1}_{\beta-1}\mathbf{A}_{\beta-1}) - \mathbf{g}^{+}_{\alpha(\beta-1)}\mathbf{C}^{-1}_{\beta-1}\mathbf{g}_{\beta} - \mathbf{g}^{+}_{\alpha}\mathbf{C}^{-1}_{\alpha-1}\mathbf{g}_{\beta(\alpha-1)} + C_{\alpha\beta}. \quad (39)$$

In Eq. (39) it is taken into account that $\operatorname{cov}(\mathbf{A}_n, \mathbf{A}_n) = \mathbf{C}_n$ and the following notations are introduced:

$$\mathbf{g}_{\alpha(\beta-1)} = (C_{\alpha 1}, \ldots, C_{\alpha(\beta-1)}), \qquad \mathbf{g}_{\beta(\alpha-1)} = (C_{1\beta}, \ldots, C_{(\alpha-1)\beta}),$$
$$(\mathbf{C}^{-1}_{(\alpha-1)\,(\beta-1)})_{ij} = \begin{cases} (\mathbf{C}^{-1}_{\alpha-1})_{ij}, & i \le \alpha-1, \quad j \le \alpha-1; \\ 0, & \alpha-1 < i \le \beta-1, \quad \alpha-1 < j \le \beta-1, \end{cases} \quad (40)$$

where $i = 1, \ldots, \beta-1$ and $j = 1, \ldots, \beta-1$.

It can be shown that the relationship $\operatorname{cov}(\tilde{\mathbf{Q}}^{+}\mathbf{Y}, \tilde{\mathbf{U}}^{+}\mathbf{Y}) = \tilde{\mathbf{Q}}^{+}\mathbf{R}\tilde{\mathbf{U}}$, where $\mathbf{R} = \operatorname{cov}(\mathbf{Y}, \mathbf{Y})$ is the matrix of the correlation moments (covariance matrix) of the vector $\mathbf{Y}$, is valid for all deterministic vectors $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{U}}$ and the random vector $\mathbf{Y}$. Calculating covariance (39) with the help of the above relationship and allowing for notations (40), we obtain

$$\begin{aligned} &\operatorname{cov}(\mathbf{g}^{+}_{\alpha(\beta-1)}\mathbf{C}^{-1}_{(\alpha-1)\,(\beta-1)}\mathbf{A}_{\beta-1}, \mathbf{g}^{+}_{\beta}\mathbf{C}^{-1}_{\beta-1}\mathbf{A}_{\beta-1}) - \mathbf{g}^{+}_{\alpha(\beta-1)}\mathbf{C}^{-1}_{\beta-1}\mathbf{g}_{\beta} - \mathbf{g}^{+}_{\alpha}\mathbf{C}^{-1}_{\alpha-1}\mathbf{g}_{\beta(\alpha-1)} + C_{\alpha\beta} \\ &\quad = \mathbf{g}^{+}_{\alpha}\mathbf{C}^{-1}_{\alpha-1}\mathbf{g}_{\beta(\alpha-1)} - \mathbf{g}^{+}_{\alpha}\mathbf{C}^{-1}_{\alpha-1}\mathbf{g}_{\beta(\alpha-1)} - \mathbf{g}^{+}_{\alpha(\beta-1)}\mathbf{C}^{-1}_{\beta-1}\mathbf{g}_{\beta} + C_{\alpha\beta} \\ &\quad = C_{\alpha\beta} - \mathbf{g}^{+}_{\alpha(\beta-1)}\mathbf{C}^{-1}_{\beta-1}\mathbf{g}_{\beta} = C_{\alpha\beta} - C_{\alpha\beta} = 0. \quad (41) \end{aligned}$$

According to Eq. (41), the random quantities B_α and B_β are uncorrelated for all $\beta \ne \alpha$. Therefore, the random quantities $\{B_i\}_{i=1}^{N}$, which satisfy Eq. (33), are mutually uncorrelated.

We denote the variance of the quantity B_n as $D(B_n) = \operatorname{cov}(B_n, B_n)$. Obviously, $D(B_1) = 1$ and for $n > 1$ we write

$$\begin{aligned} D(B_n) &= D\Big[\gamma_n\,(A_n - \mathbf{A}^{+}_{n-1}\mathbf{C}^{-1}_{n-1}\mathbf{g}_n)/\sqrt{H}\Big] \\ &= \left[D(\mathbf{A}^{+}_{n-1}\mathbf{C}^{-1}_{n-1}\mathbf{g}_n) + D(A_n) - 2\operatorname{cov}(\mathbf{A}^{+}_{n-1}\mathbf{C}^{-1}_{n-1}\mathbf{g}_n, A_n)\right]/H = 1. \end{aligned}$$

Therefore, the random quantities of the set $\{B_n\}_{n=1}^{N}$ have unit variance.

Note that in Statement 1 and its proof, the random quantities A_n can be interpreted as the vectors in the linear vector space, while the covariance can be interpreted as the scalar product of the vectors in this space. In fact, the covariance properties of the random quantities, which are used in the proof of Statement 1, also hold for the scalar product of the vectors. In such an interpretation, the vectors B_n in Eq. (34) specify the orthogonalization procedure for the vectors A_n, which is similar to the Gram–Schmidt procedure.

Let $\mathbf{U}$ be an invertible $N \times N$ matrix and $\mathbf{E}$ be a column vector with dimension N. Assume that the matrix $\mathbf{W}$ and the vector $\mathbf{R}$ are obtained from the matrix $\mathbf{U}$ and the vector $\mathbf{E}$ as follows:

$$W_{ij} = \begin{cases} U_{gj}, & i = q; \\ U_{qj}, & i = g; \\ U_{ig}, & j = q; \\ U_{iq}, & j = g; \\ U_{ij}, & i \ne g \wedge i \ne q \wedge j \ne g \wedge j \ne q, \end{cases} \qquad R_i = \begin{cases} E_g, & i = q; \\ E_q, & i = g; \\ E_i, & i \ne g \wedge i \ne q. \end{cases}$$

It can be shown that the matrix $\mathbf{W}$ is invertible and the following equality takes place:

$$\mathbf{E}^{+}\mathbf{U}^{-1}\mathbf{E} = \mathbf{R}^{+}\mathbf{W}^{-1}\mathbf{R}. \quad (42)$$

Successively transposing the rows and columns in the matrix $\mathbf{G}^{-1}_{\nu}$ in Eq. (25), as well as the elements of the vector $\mathbf{X}_\nu$ in Eq. (25), and using Eq. (42), we obtain the relationships

$$\begin{pmatrix}\mathbf{X}_{ci}\\ \mathbf{X}_{si}\end{pmatrix}^+ \begin{pmatrix}\mathbf{G}_{ci} & \mathbf{G}_{csi}\\ \mathbf{G}_{sci} & \mathbf{G}_{si}\end{pmatrix}^{-1} \begin{pmatrix}\mathbf{X}_{ci}\\ \mathbf{X}_{si}\end{pmatrix} = \mathbf{Y}_i^+\mathbf{F}_i^{-1}\mathbf{Y}_i,$$

$$\begin{pmatrix}\mathbf{X}_{ci}\\ \mathbf{X}_{s(i-1)}\end{pmatrix}^+ \begin{pmatrix}\mathbf{G}_{ci} & \mathbf{G}_{csi(i-1)}\\ \mathbf{G}_{sc(i-1)i} & \mathbf{G}_{s(i-1)}\end{pmatrix}^{-1} \begin{pmatrix}\mathbf{X}_{ci}\\ \mathbf{X}_{s(i-1)}\end{pmatrix} = \mathbf{Y}_{i-1}^+\mathbf{F}_{i-1}^{-1}\mathbf{Y}_{i-1},$$

$$\begin{pmatrix}\mathbf{X}_{c(i-1)}\\ \mathbf{X}_{s(i-1)}\end{pmatrix}^+ \begin{pmatrix}\mathbf{G}_{c(i-1)} & \mathbf{G}_{cs(i-1)}\\ \mathbf{G}_{sc(i-1)} & \mathbf{G}_{s(i-1)}\end{pmatrix}^{-1} \begin{pmatrix}\mathbf{X}_{c(i-1)}\\ \mathbf{X}_{s(i-1)}\end{pmatrix} = \mathbf{Y}_{i-2}^+\mathbf{F}_{i-2}^{-1}\mathbf{Y}_{i-2}.$$

where $Y_k = X_{ck}$ for even k, $Y_k = X_{sk}$ for odd k, $\mathbf{Y}_i = (Y_1, \ldots, Y_i)$, and $\mathbf{F}_i = \mathrm{cov}(\mathbf{Y}_i, \mathbf{Y}_i)$. Hence, the quantities l_{si}^2, l_{ci}^2 ($i = 1, \ldots, \nu_{\max}$) can be represented in the form

$$l_{si}^2 = \mathbf{Y}_i^+\mathbf{F}_i^{-1}\mathbf{Y}_i - \mathbf{Y}_{i-1}^+\mathbf{F}_{i-1}^{-1}\mathbf{Y}_{i-1}, \qquad l_{ci}^2 = \mathbf{Y}_{i-1}^+\mathbf{F}_{i-1}^{-1}\mathbf{Y}_{i-1} - \mathbf{Y}_{i-2}^+\mathbf{F}_{i-2}^{-1}\mathbf{Y}_{i-2}. \tag{43}$$

It is shown in Statement 1 that the random quantities B_i^2 are the squared Gaussian random quantities B_i. By analogy, using Eq. (43), it can be shown that the random quantities l_{ci}^2 and l_{si}^2 are also the squared Gaussian random quantities l_{ci} and l_{si}, respectively. Let us indicate the properties of the Gaussian random quantities of the sets $\{l_{ci}\}_{i=1}^{\nu_{\max}}$ and $\{l_{si}\}_{i=1}^{\nu_{\max}}$.

Statement 2. The Gaussian random quantities of the set $\{l_{ci}\}_{i=1}^{\nu_{\max}} \cup \{l_{si}\}_{i=1}^{\nu_{\max}}$ are independent and have unit variances, and, in addition, the mathematical expectations of the random quantities of the set $\{l_{ci}\}_{i=\nu_0+1}^{\nu_{\max}} \cup \{l_{si}\}_{i=\nu_0+1}^{\nu_{\max}}$ are equal to zero.

The proof of the Statement 2 immediately follows from Statement 1 and Eq. (43).

4. CHARACFTERISTICS OF THE ESTIMATES OF THE NUMBER OF RADIO SIGNALS WITH UNKNOWN AMPLITUDES AND PHASES

Let us analyze algorithms (28)–(31) in terms of the abridged error probability. To this end, the explicit form of realization (2) of the received data is substituted into Eq. (27) for the modified maximized LRF logarithm. On the basis of Eq. (32) and Statement 1, Eq. (26) for the maximized LRF logarithm can be rewritten as

$$L_{\mathrm{m}}(\nu) = \frac{1}{2}\sum_{i=1}^{\nu} Q_i, \tag{44}$$

where

$$Q_i = \begin{cases}(d_{ci} + \xi_{ci})^2 + (d_{si} + \xi_{si})^2, & i \le \nu_0;\\ \xi_{ci}^2 + \xi_{si}^2, & i > \nu_0,\end{cases}$$

$i = 1, \ldots, \nu_{\max}$, d_{ci} and d_{si} are the mathematical expectations of the random quantities l_{ci} and l_{si}, respectively, and ξ_{ci} and ξ_{si} are the independent Gaussian random quantities with zero mathematical expectations and unit variances. According to Eq. (44), the LRF logarithm in Eq. (26) is a nondecreasing function of the number of signals. In addition, it can be shown that for any $i \le \nu_0$, the quantity $d_i^2 = d_{ci}^2 + d_{si}^2$ monotonical increases with increasing SNR z_i^2.

Using Eq. (44), the modified maximized LRF logarithm (28) with a linear penalty function can be rewritten in the form

$$L_{\mathrm{D}}(\nu; \mathbf{X}_\nu, \kappa) = \frac{1}{2}\sum_{i=1}^{\nu} Q_i - \kappa\nu. \tag{45}$$

Then, using Eq. (45), one can calculate the abridged error probability (13) for algorithm (28) with a linear penalty function as

$$p_{\mathrm{t}1} = 1 - p(Q_{\nu_0} > 2\kappa, Q_{\nu_0+1} < 2\kappa) = 1 - F_{\nu_0+1}(2\kappa) + F_{\nu_0}(2\kappa)F_{\nu_0+1}(2\kappa), \tag{46}$$

where $F_{\nu_0}(x)$ is the function of the noncentral χ^2 distribution with two degrees of freedom and the noncentrality parameter $d^2_{\nu_0} = d^2_{\mathrm{c}\nu_0} + d^2_{\mathrm{s}\nu_0}$, and $F_{\nu_0+1}(x)$ is the function of the central χ^2 distribution with two degrees of freedom. With allowance for the properties of the function $F_{\nu_0}(x)$, it can be shown that for an unlimited increase in the parameter $d^2_{\nu_0}$, the abridged error probability is written as

$$p_{\mathrm{t}1} \to 1 - F_{\nu_0+1}(2\kappa). \tag{47}$$

Therefore, the abridged error probability tends to a constant value with increasing parameter $d^2_{\nu_0}$. The limiting value of error probability (47) can be used for approximate choice of the coefficient κ in Eq. (28). Indeed, the choice of κ can be recommended by the value of the acceptable error probability $p_{\mathrm{t}1}$.

As is noted above, the abridged error probability is the lower boundary of the total error probability. Therefore, it follows from Eqs. (44) and (47) that the total error probability does not tend to zero with increasing $d^2_{\nu_0}$ and, hence, SNR $z^2_{\nu_0}$. This property is a serious disadvantage of algorithm (28).

Unfortunately, in the general case, it is difficult to analytically obtain even the abridged error probability for algorithm (29). This probability can be calculated if the radio signals are orthogonal, i.e., algorithms (29) and (30) coincide. When estimating the number of correlated radio signals, the error probability for algorithm (29) can be obtained using statistical simulation.

Let us analyze algorithm (30) with the invariant random penalty function (30). For this purpose, we again use Eq. (44) to represent the modified maximized LRF logarithm with invariant random penalty function (30) in the form

$$L_{\mathrm{D}2}(\nu; \mathbf{X}_\nu, \kappa_2) = \frac{1}{2}\sum_{i=1}^{\nu} Q_i - \frac{1}{2}\kappa_2 \nu \max_i Q_i, \qquad i = 1, \ldots, \nu_{\max}. \tag{48}$$

Calculate the abridged error probability in Eq. (13) as

$$p_{\mathrm{t}2} = 1 - p\big(Q_{\nu_0} > \kappa_2 \max_i Q_i, Q_{\nu_0+1} < \kappa_2 \max_i Q_i\big), \qquad i = 1, \ldots, \nu_{\max}. \tag{49}$$

An auxiliary statement can formulated for the further calculation of the probability $p_{\mathrm{t}2}$.

Let $\{\tilde{A}_i\}_{i=1}^{M}$ be a set of the mutually independent random quantities; $k = 1, \ldots, M$; $0 \le h \le 1$; and $\tilde{B} = \max_i \tilde{A}_i$, where $i = 1, \ldots, M$, $i \ne k$ and $i \ne k+1$. Then

$$p(\tilde{A}_k > h \max_{i=1,\ldots,M} \tilde{A}_i, \tilde{A}_{k+1} < h \max_{i=1,\ldots,M} \tilde{A}_i) = p(\tilde{A}_k > h\tilde{A}_{k+1}, \tilde{A}_k > h\tilde{B}) \\ - p(\tilde{A}_k > \tilde{A}_{k+1}, \tilde{A}_{k+1} > h\tilde{A}_k, \tilde{A}_{k+1} > h\tilde{B}) - p(\tilde{A}_{k+1} > \tilde{A}_k, \tilde{A}_k > h\tilde{A}_{k+1}, \tilde{A}_k > h\tilde{B}). \tag{50}$$

Using Eqs. (49) and (50), we can write the final formula for the abridged error probability for algorithm (30) with the invariant penalty function:

$$p_{\mathrm{t}2} = 1 - \int_0^\infty W_{\nu_0}(x) F_{\nu_0+1}\left(\frac{x}{\kappa_2}\right) F_{Q_{\max}}\left(\frac{x}{\kappa_2}\right) \mathrm{d}x + \int_0^\infty W_{\nu_0+1}(x) F_{Q_{\max}}\left(\frac{x}{\kappa_2}\right) \\ \times \left[F_{\nu_0}\left(\frac{x}{\kappa_2}\right) - F_{\nu_0}(x)\right] \mathrm{d}x + \int_0^\infty W_{\nu_0}(x) F_{Q_{\max}}\left(\frac{x}{\kappa_2}\right) \left[F_{\nu_0+1}\left(\frac{x}{\kappa_2}\right) - F_{\nu_0+1}(x)\right] \mathrm{d}x, \tag{51}$$

where $W_i(x)$ and $F_i(x)$ are the probability density and the distribution function of the random quantity Q_i, respectively, while $W_{Q_{\max}}(x)$ and $F_{Q_{\max}}(x)$ are the probability density and the distribution function of the random quantity $Q_{\max} = \max_i Q_i$, respectively, where $i = 1, \ldots, \nu_{\max}$, $i \neq \nu_0$, and $i \neq \nu_0 + 1$.

The distribution function $F_{Q_{\max}}(x)$ is written in the form

$$F_{Q_{\max}}(x) = \prod_{\substack{i=1,\\ i\neq\nu_0, i\neq\nu_0+1}}^{\nu_{\max}} F_i(x).$$

Hereafter, for $i \leq \nu_0$, the dependences $W_i(x)$ and $F_i(x)$ are the probability density and the distribution function of the noncentral χ^2 distribution with the noncentrality parameter $d_i^2 = d_{\mathrm{c}i}^2 + d_{\mathrm{s}i}^2$ and two degrees of freedom, respectively [14], while for $i > \nu_0$, the above dependences are the probability density and the distribution function of the central χ^2 distribution with two degrees of freedom, respectively [14].

Using again Eq. (44), we rewrite the modified maximized LRF logarithm (31) with an inverse penalty function in the form

$$L_{\mathrm{B}}(\nu; \mathbf{X}_\nu, n) = \left(\frac{1}{2}\sum_{i=1}^{\nu} Q_i\right)^n \Big/ \nu. \tag{52}$$

Using Eq. (52), we can calculate the abridged error probability (13) for the algorithm with the inverse penalty function as

$$p_{\mathrm{t}3} = 1 - \int_0^\infty W_{\nu_0+1}(y) \int_0^\infty W_{\nu_0}(x) \left[F_0(x/A^*) - F_0(y/B^* - x)\right] \mathrm{d}x\, \mathrm{d}y. \tag{53}$$

Here, $F_0(x)$ is the function of the noncentral χ^2 distribution with the noncentrality parameter $\sum_{i=1}^{\nu_0-1}(d_{\mathrm{c}i}^2+d_{\mathrm{s}i}^2)$ and $2\,(\nu_0 - 1)$ degrees of freedom. As above, $W_{\nu_0}(x)$ is the probability density function of the noncentral χ^2 distribution with two degrees of freedom and the noncentrality parameter $d_i^2 = d_{\mathrm{c}i}^2 + d_{\mathrm{s}i}^2$, $W_{\nu_0+1}(x)$ is the probability density function of the central χ^2 distribution with two degrees of freedom [14], $A^* = \sqrt[n]{\nu_0/(\nu_0 - 1)} - 1$, and $B^* = \sqrt[n]{(\nu_0 + 1)/\nu_0} - 1$.

Let us study the dependence of the abridged error probability of algorithms (30) and (31) on the SNR. For better clarity, it is assumed that $a_i = a_0$ for all i, $K_{ij} = E$ for $i = j$, and $K_{ij} = 0$ for $i \neq j$. Then for any i, the quantities $d_{\mathrm{c}i}$ and $d_{\mathrm{s}i}$ are only functions of the SNR $z = a_0\sqrt{2E/N_0}$ and can be written as

$$d_{\mathrm{c}i} = d_{\mathrm{s}i} = z/\sqrt{2}. \tag{54}$$

In addition, in this case, the abridged error probability for algorithm (29) coincides with probability (51).

To study algorithms (30) and (31), we performed numerical calculations using the obtained analytical formulas given by Eqs. (51) and (53) and the statistical simulation of the algorithms. Using the results of numerical calculations with the help of Eqs. (51) and (53), we have established that the abridged error probability of algorithms (30) and (31) significantly depends on the parameters κ_2 and n, respectively. Therefore, it is proposed to seek the studied-algorithm parameters that are optimal from the viewpoint of the minimum abridged error probability. To this end, it is sufficient to substitute the values of d_{ν_0} and d_{ν_0-1} into Eqs. (51) and (53) for any particular values of the SNR vector $\mathbf{z}_{\nu_{\max}}$ and then find the minima of abridged error probabilities (51) and (53) as functions of the corresponding parameter (κ_1 and n for the probabilities $p_{\mathrm{t}2}$ and $p_{\mathrm{t}3}$, respectively).

The considered procedure was performed for the above-described case (54). Then we obtained the quantities $\kappa_2^{\mathrm{opt}} = 0.31$ and $n^{\mathrm{opt}} = 2.5$ such that $p_{\mathrm{t}2}(\kappa_2^{\mathrm{opt}}) = \inf p_{\mathrm{t}2}(\kappa_2)$ and $p_{\mathrm{t}3}(n^{\mathrm{opt}}) = \inf p_{\mathrm{t}3}(n)$. The obtained values are accepted as the parameters of algorithms (30) and (31) during the statistical simulation and numerical calculations. It should be noted that the above optimization makes sense only if the optimal value of the algorithm parameter weakly depends on the SNR.

During the study, it was clarified that the optimal values of the parameters κ_1 and κ_2 in algorithms (29) and (30), respectively, significantly depend only on the maximum possible number $\nu_{\max}$ of signals (if the signals are orthogonal). At the same time, the optimal values of the parameter n in Eq. (31) are almost identical for all SNR values and $\nu_{\max} \leq 21$.

Figure 2 shows the results of calculating the abridged error probability of algorithms (30) and (31) by Eqs. (51) and (53). The solid line shows the error probability in determining the number ν_0 of radio signals with known amplitudes and phases by algorithm (8) as a function of the SNR. The dotted line shows the theoretical dependence of the error probability in determining the value of ν_0 by algorithm (30) on the SNR (see Eq. (51)). The dashed line shows the theoretical dependence of the error probability in evaluating ν_0 by algorithm (31) on the SNR (see Eq. (53)). The squares and circles show the error probabilities in evaluating ν_0 by algorithms (30) and (31), respectively, on the SNR. The latter values are obtained using statistical simulation. Algorithm (28) with linear penalty function (28) is not shown in Fig. 2 since its optimal value of the parameter κ significantly depends on the SNR.

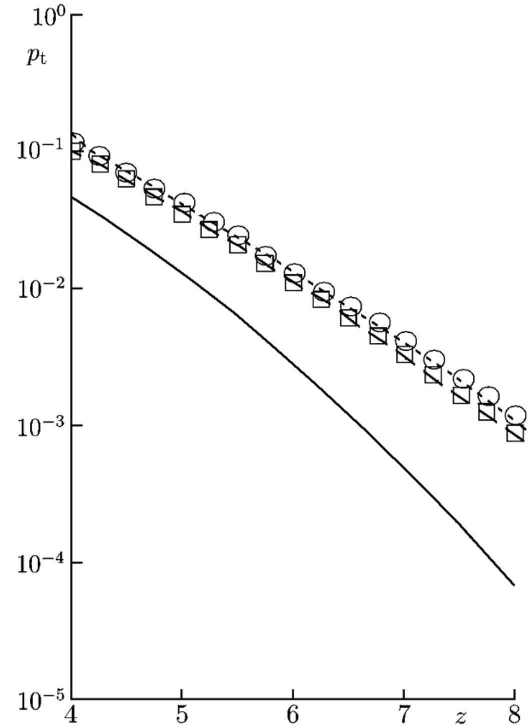

Fig. 2. Error probability for the *a priori* unknown amplitudes and phases.

The calculation results in Fig. 2 indicate that the formulas obtained for the abridged error probabilities are in satisfactory agreement with the data of the statistical simulation of the algorithms for estimating the number of signals. Figure 2 also shows that the *a priori* ignorance of the amplitudes and phases of the received signals substantially affects the accuracy of their number estimation. It can also be noted that the operation quality of optimized algorithms (30) and (31) is almost the same.

5. CONCLUSIONS

The proposed abridged error probability in estimating the number of radio signals allows one to relatively simply characterize the efficiency of various algorithms for estimating the number of radio signals. The obtained results of analyzing several algorithms for estimating the number of signals make it possible to reasonably choose the required estimation algorithm and optimize its parameters. The results of theoretical calculation are in satisfactory agreement with those of statistical simulation. It has been shown that *a priori* ignorance of the amplitudes and phases of the received radio signals can significantly deteriorate the operation quality of the algorithms for estimating the number of signals.

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