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Determining the Number of Radio Signals with Unknown Phases

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Abstract – *We have carried out the synthesis and analysis of the maximum likelihood and quasi-likelihood algorithms in order to determine the number of radio signals, introducing and calculating the abridged probability of error in the assessment of the number of signals as the quantitative characteristic for the performance evaluation of these algorithms. The achieved analytical results lead us to the comparison between the characteristics of maximum likelihood and quasi-likelihood algorithms in determining the number of signals. We also consider the possibility for the application of the quasi-likelihood algorithm for the determination of the number of radio signals with partially unknown initial phases.* ***Copyright © 2015 Praise Worthy Prize S.r.l. - All rights reserved.***

Keywords*:* *Maximum Likelihood Estimation, Estimation of the Number of Signals, Error Probability, Signal with Unknown Phase, Quasi-Likelihood Estimation*

I. Introduction

Dealing with information transfer and processing, we are often concerned with the problem of determining the number of the received signals. Thus, in case of multipath radio channel operation in MIMO systems [1], [2], the number of rays is, as a rule, a priori unknown, and it is to be determined. Also, for radar and acoustic radar (either active, or passive) sensing, it is a common situation when the number of signal sources located by the antenna array is unknown [3]-[6]. However, despite its occurrence, the problem of the determination of the number of the received signals is still only partially solved. In many cases difficulties arise while evaluating the determining algorithm structure. Practically, there are no results yet of the theoretical analysis of the performance quality evaluation concerning the algorithms determining the number of signals.

Moreover, there is no universally adopted proper quantitative characteristic for the evaluation of the efficiency of such algorithms.

In the absence of the quantitative characteristics for the algorithms determining the number of signals, the comparison between such algorithms and the choice of the most efficient of them are highly problematic.

In paper [7], there are presented the modifications of the maximum likelihood method that provide us with the algorithm for the estimation of the number of signals with unknown amplitudes and its thorough study.

Then, in study [8] the complicated algorithm is introduced, also based on the modifications of the maximum likelihood method. It is intended for the estimation of the number of radio signals with unknown amplitudes and phases. In [9] it has done a general study of determining the number of signals with known amplitudes and unknown non-energy parameters.

Besides, for all algorithms synthesized in these researches, their basic characteristics are calculated. For the estimation of the number of signals in [7], [8], the modifications of the maximum likelihood method are used, and not the classical form of it, because one of the unknown signal parameters in question is amplitude.

Indeed, as it is shown in [7], [8], in this very case the maximum likelihood method turns out to be inconsistent.

But here we consider the situation when only the initial phases of the observable radio signals are unknown, and thus the maximum likelihood method tends to be applicable for estimating the number of signals. Besides the maximum likelihood estimation, we produce and study the quasi-likelihood estimation [10], as in practice the signal parameters can be only partially unknown. Under partially unknown signal parameter we mean the parameter for which true value is not exactly known, but some limited interval possessing this value can be specified. In other words, partially unknown signal parameter is identified only as belonging to some limited prior interval. Signal parameter value set with some known finite error is the example.

In order to obtain the maximum likelihood estimation of the number of signals with unknown parameters, we should substitute the maximum likelihood estimates of these unknown parameters into the logarithm of the functional of likelihood ratio (FLR), at first. In case of estimating of the number of signals with partially unknown parameters, we suggest the quasi-likelihood algorithm [10].

For quasi-likelihood algorithm we substitute some expected values of these parameters from the appropriate prior intervals of their allowed values into the logarithm of FLR, instead of the values of partially unknown parameters.

Below we carry out the synthesis and analysis of the quasi-likelihood algorithm for the estimation of the number of radio signals with partially unknown phases against Gaussian white noise.

We also conduct the synthesis and analysis of the maximum likelihood algorithm for estimating the number of signals with completely unknown phases possessing values within the prior interval 2π by length.

Efficiency of each considered algorithm we will characterize by the probability of error in estimating the number of signals.

For its definition we will use the abridged error probability [8].

II. The Synthesis and Analysis of the Quasi-Likelihood Algorithm for the Estimation of the Number of Radio Signals

Let us suppose that over time interval $[0,T]$ the sum of ν possible narrowband radio signals $s_i(t,\varphi_i) = a_i f_i(t) cos(\omega_i t + \Psi_i(t) - \varphi_i)$ can be observed, so the set of signals:

$$s_i(t,\nu,\varphi_\nu) = \sum_{i=1}^{\nu} s_i(t,\varphi_i) = \sum_{i=1}^{\nu} a_i f_i(t) cos(\omega_i t + \Psi_i(t) - \varphi_i) \quad (1)$$

can be passed to the measurer input.

In Eq. (1) it is designated: $a_i, \omega_i \in R^1$ are amplitude and frequency, $\varphi_i \in [0,2\pi]$ is possible phase, $\Psi_i(t) \in L_1(0,T)$ is phase modulating function, $f_i(t) \in L_2(0,T)$ is envelope of the *i*-th signal, and $\varphi_\nu = \|\varphi_i\|_{i=1}^{\nu}$, $\nu = \overline{1,\nu_{max}}$.

We designate the true number of signals as ν_0. We also presuppose that the signal (1) be received against additive Gaussian white noise $n(t)$ with one-sided spectral density N_0.

Then, the following realization:

$$x(t) = n(t) + \sum_{i=1}^{\nu_0} a_i f_i(t) cos(\omega_i t + \Psi_i(t) - \varphi_{0i}) \quad (2)$$

is accessible for the processing. Here φ_{0i}, $i = \overline{1,\nu_0}$ are true signal phase values.

In [11] the formula is made of the logarithm of FLR for arbitrary signal $s(t,l)$ containing unknown parameters *l*, when additive Gaussian white noise is an interference:

$$L(l) = \frac{2}{N_0}\int_0^T x(t)\, s(t,l)\, dt - \frac{1}{N_0}\int_0^T s^2(t,l)\, dt \quad (3)$$

After substituting Eq. (1) in Eq. (3), we then rewrite the logarithm of FLR for the set of signals (1) as:

$$L(\nu,\varphi_\nu) = \frac{2}{N_0}\sum_{i=1}^{\nu} a_i \int_0^T x(t) f_i(t) cos(\omega_i t + \Psi_i(t) - \varphi_i) dt + \\ - \frac{1}{N_0}\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} a_i a_j \int_0^T f_i(t) f_j(t) cos(\omega_i t + \Psi_i(t) - \varphi_i) \times \\ \times cos(\omega_j t + \Psi_j(t) - \varphi_j) dt$$

Further, we present the latter expression in this form:

$$L(\nu,\varphi_\nu) = \\ = \frac{2}{N_0}\sum_{i=1}^{\nu} a_i \int_0^T x(t) f_i(t) cos(\omega_i t + \Psi_i(t) - \varphi_i) dt + \quad (4) \\ - \frac{1}{N_0}\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} a_i a_j K_{ij}$$

where $\nu \in \overline{1,\nu_{max}}$, and K_{ij} is dot product of radio signals $s_i(t,\varphi_i)$ and $s_j(t,\varphi_j)$, so:

$$K_{ij} = \frac{1}{2}\int_0^T f_i(t) f_j(t) cos(\omega_i t + \Psi_i(t) - \varphi_i + \\ - \omega_j t - \Psi_j(t) + \varphi_j) dt + \\ + \frac{1}{2}\int_0^T f_i(t) f_j(t) cos(\omega_i t + \Psi_i(t) - \varphi_i + \\ + \omega_j t + \Psi_j(t) - \varphi_j) dt \quad (5)$$

Now we are to consider the case when the signals in Eq. (1) satisfy to the bandlimitedness condition for all $i = \overline{1,\nu_{max}}$ [11]:

$$\omega_i / \Delta\omega_i >> 1 \quad (6)$$

Here $\Delta\omega_i$ is *i*-th signal bandwidth. For this case the second summands in Eq. (5) are small in comparison with the first ones. It allows rewriting the expression (5) into:

$$K_{ij} = V_{cij}\, cos(\varphi_i - \varphi_j) + V_{sij}\, sin(\varphi_i - \varphi_j) \quad (7)$$

where:

$$V_{cij} = \frac{1}{2}\int_0^T f_i(t) f_j(t) cos((\omega_i - \omega_j)\, t + \Psi_i(t) - \Psi_j(t)) dt$$

$$V_{sij} = \frac{1}{2}\int_0^T f_i(t)\, f_j(t)\ sin\left((\omega_i - \omega_j)\ t + \Psi_i(t) - \Psi_j(t)\right)\, dt$$

After normalizing Eq. (7), we obtain the correlation coefficient of radio signals:

$$\rho_{ij} = K_{ij} \big/ \sqrt{E_i E_j} = \rho_{cij}\, cos\left(\varphi_i - \varphi_j\right) + \\ + \rho_{sij}\, sin\left(\varphi_i - \varphi_j\right) \quad (8)$$

Here $\rho_{cij} = V_{cij} \big/ \sqrt{E_i E_j}$, $\rho_{sij} = V_{sij} \big/ \sqrt{E_i E_j}$, and $E_i = K_{ii}$ is the ***i***-th signal energy. We presuppose that radio signals (1) phases are partially unknown, i.e., there are only known the final prior intervals including the all possible values of the phases. While synthesizing the estimation algorithm of the number of signals, we will change the unknown values of phases $\left\|\varphi_i\right\|_{i=1}^{\nu_{max}}$ in Eq. (4) by their expected values $\left\|\varphi_i^*\right\|_{i=1}^{\nu_{max}}$ from the specified prior intervals:

$$L\left(\nu, \varphi_\nu^*\right) = L(\nu) = \\ = \frac{2}{N_0}\sum_{i=1}^{\nu} a_i \int_0^T x(t)\, f_i(t) cos\left(\omega_i t + \Psi_i(t) - \varphi_i^*\right) dt + \\ - \frac{1}{N_0}\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} a_i a_j K_{ij}^* \quad (9)$$

where:

$$K_{ij}^* = V_{cij}\, cos\left(\varphi_i^* - \varphi_j^*\right) + V_{sij}\, sin\left(\varphi_i^* - \varphi_j^*\right)$$

$$\varphi_\nu^* = \left\|\varphi_i^*\right\|_{i=1}^{\nu}$$

Using the decision statistics (9) we can write down the quasi-likelihood estimation algorithm of the number of signals in Eq. (2):

$$\hat{\nu} = arg\,\sup_{\nu} L(\nu),\ \ \nu = \overline{1, \nu_{max}} \quad (10)$$

Let us consider the properties of the decision statistics (9). For this purpose we substitute the realization of the observable data (2) in Eq. (9):

$$L(\nu) = \sum_{j=1}^{\nu} z_j \sum_{i=1}^{\nu_0} z_i \Big[\ \rho_{cij}\, cos\left(\varphi_{0i} - \varphi_{0j} - \Delta_j\right) + \\ + \rho_{sij}\, sin\left(\varphi_{0i} - \varphi_{0j} - \Delta_j\right)\ \Big] + \sum_{j=1}^{\nu} z_j \xi_j + \\ - \frac{1}{2}\sum_{j=1}^{\nu}\sum_{i=1}^{\nu} z_i z_j \Big[\ \rho_{cij}\, cos\left(\varphi_{0i} - \varphi_{0j} + \Delta_i - \Delta_j\right) + \\ + \rho_{sij}\, sin\left(\varphi_{0i} - \varphi_{0j} + \Delta_i - \Delta_j\right)\Big] \quad (11)$$

Here:

$$\xi_i = \sqrt{2/N_0 E_i}\int_0^T n(t)\, f_i(t) cos\left(\omega_i t + \Psi_i(t) - \varphi_i^*\right)\, dt$$

are Gaussian random variables with zero mathematical expectations and unit dispersions, $\Delta_i = \varphi_i^* - \varphi_{0i}$ is a parameter characterizing the deviation of the expected initial phases φ_i^* from their true values φ_{0i}, $z_i^2 = 2a_{0i}^2 E_i \big/ N_0$ is a signal-to-noise ratio (SNR) for ***i***-th signal in Eq. (2).

Efficiency of algorithm estimating the number of signals can be described by the error probability $p_e = p\left(\hat{\nu} \neq \nu_0\right)$. However, the calculation of this probability requires considerable computational resources. In order to obtain the simplified approximate formula for the error probability, we must have in mind that the any algorithm $\Re$ estimating the number of signals can be presented as:

$$\hat{\nu} = arg\,\sup_{\nu} R\left(\nu; x(t)\right)$$

where $R\left(\nu; x(t)\right)$ is the functional defined by the structure of the algorithm $\Re$ and depending upon the number of signals and the realization of the observable data. Accordingly, the error probability for the algorithm $\Re$ can be written down in the kind of:

$$p_e = 1 - p\left[\begin{matrix} R\left(\nu_0; x(t)\right) > R\left(i; x(t)\right) \\ i \neq \nu_0, i = \overline{1, \nu_{max}} \end{matrix}\right] \quad (12)$$

Now, as approximation for the error probability, we introduce the abridged error probability p_a for the algorithm $\Re$ defined by the relation:

$$p_a = 1 - p\Big[R\left(\nu_0; x(t)\right) > R\left(\nu_0 + 1; x(t)\right) \\ R\left(\nu_0; x(t)\right) > R\left(\nu_0 - 1; x(t)\right)\Big] \quad (13)$$

From the definition (13) follows that the abridged error probability is the lower bound for the error probability (12), when $1 < \nu_0 < \nu_{max}$. Also it should be noted that the abridged error probability coincides with the error probability in case when $\nu_{max} = 3$ and $\nu_0 = 2$.

In terms of the algorithm (10), we can rewrite Eq. (13) as:

$$p_a = 1 - p\left[L(\nu_0) > L(\nu_0 + 1), L(\nu_0) > L(\nu_0 - 1)\right] \quad (14)$$

Then, from formulas (11) and (14), we obtain the following presentation for the abridged error probability p_a in the algorithm (10):

$$p_a = 1 - p\left(\xi_{\nu_0} > -R, \xi_{\nu_0+1} < Q\right) \tag{15}$$

where:

$$\begin{aligned}
R = \frac{1}{2} z_{\nu_0} \rho_{c\nu_0\nu_0} - \sum_{i=1}^{\nu_0} z_i \Big[\rho_{ci\nu_0} cos\left(\varphi_{0i} - \varphi_{0\nu_0} - \Delta_{\nu_0}\right) + \\
+ \rho_{si\nu_0} sin\left(\varphi_{0i} - \varphi_{0\nu_0} - \Delta_{\nu_0}\right) \Big] + \\
+ \sum_{i=1}^{\nu_0 - 1} z_i \Big[\rho_{ci\nu_0} cos\left(\varphi_{0i} - \varphi_{0\nu_0} + \Delta_i - \Delta_{\nu_0}\right) + \\
+ \rho_{si\nu_0} sin\left(\varphi_{0i} - \varphi_{0\nu_0} + \Delta_i - \Delta_{\nu_0}\right) \Big] \\
Q = \frac{1}{2} z_{\nu_0+1} \rho_{c(\nu_0+1)(\nu_0+1)} + \\
- \sum_{i=1}^{\nu_0} z_i \Big[\rho_{ci\nu_0+1} cos\left(\varphi_{0i} - \varphi_{0\nu_0+1} + \Delta_i - \Delta_{\nu_0+1}\right) + \\
+ \rho_{si\nu_0+1} sin\left(\varphi_{0i} - \varphi_{0\nu_0+1} + \Delta_i - \Delta_{\nu_0+1}\right) \Big] + \\
- \sum_{i=1}^{\nu_0} z_i \Big[\rho_{ci\nu_0+1} cos\left(\varphi_{0i} - \varphi_{0\nu_0+1} - \Delta_{\nu_0+1}\right) + \\
+ \rho_{si\nu_0+1} sin\left(\varphi_{0i} - \varphi_{0\nu_0+1} - \Delta_{\nu_0+1}\right) \Big]
\end{aligned} \tag{16}$$

Taking into account that ξ_{ν_0} and ξ_{ν_0+1} are Gaussian random variables with parameters $(0,1)$ and correlation coefficient $\left\langle \xi_{\nu_0} \xi_{\nu_0+1} \right\rangle = \rho_{\nu_0(\nu_0+1)}$, and referring to Eqs. (15) and (16), we now find the formula for the calculation of the abridged error probability (13) for the algorithm (10):

$$p_a = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Q} exp\left(-\frac{y^2}{2}\right) \Phi\left(\frac{R + y\rho_{\nu_0(\nu_0+1)}}{\sqrt{1 - \rho^2_{\nu_0(\nu_0+1)}}} \right) dy \tag{17}$$

here $\Phi(x) = \int_{-\infty}^{x} exp\left(-t^2/2\right) dt / \sqrt{2\pi}$ is the probability integral. Eq. (17) can be rather universally applied, as it enables us to study both the situations when all the initial phases are known with only the limited accuracy and also the situations when initial phases are all a priori exactly known for some signals presented in Eq. (1).

Assuming in Eq. (15) that $\Delta_i = 0$ for all $i = \overline{1, \nu_{max}}$, from Eq. (17) we obtain the expression for the abridged error probability for the maximum likelihood estimate of the number of radio signals with a priori exactly known initial phases shown in the following Eq. (18) [7], [8]:

$$p_a = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_{\nu_0+1}/2} exp\left(-\frac{y^2}{2}\right) \Phi\left(\frac{z_{\nu_0}/2 + y\rho_{\nu_0(\nu_0+1)}}{\sqrt{1 - \rho^2_{\nu_0(\nu_0+1)}}} \right) dy$$

III. The Synthesis and Analysis of the Maximum Likelihood Algorithm for the Estimation of the Number of Radio Signals

Let us suppose that initial signal phases in Eq. (1) are totally unknown, i.e. they can possess any values from the prior interval with the length 2π.

We impose the orthogonality condition on to functions from the set $\left\{ s_i(t, \varphi_i) \right\}_{i=1}^{\nu_{max}}$:

$$\int_0^T s_i(t, \varphi_i) s_j(t, \varphi_j) dt = \begin{cases} E_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{19}$$

Then Eq. (4) has the appearance:

$$\begin{aligned}
L_1(\nu, \varphi_\nu) = \\
= \frac{2}{N_0} \sum_{i=1}^{\nu} a_i \int_0^T x(t)\, f_i(t) cos\left(\omega_i t + \Psi_i(t) - \varphi_i\right) dt + \\
- \frac{1}{N_0} \sum_{i=1}^{\nu} a_i^2 E_i
\end{aligned} \tag{20}$$

Further, we rewrite the formula (20) as:

$$\begin{aligned}
L_1(\nu, \varphi_\nu) = \\
= \frac{2}{N_0} \sum_{i=1}^{\nu} \left[a_i X_i\, cos\varphi_i + a_i X_i\, cos\varphi_i - \frac{1}{2} a_i^2 E_i \right]
\end{aligned} \tag{21}$$

Here the following designations are introduced:

$$X_i = \int_0^T x(t)\, f_i(t) cos\left(\omega_i t + \Psi_i(t)\right) dt$$

$$Y_i = \int_0^T x(t)\, f_i(t) sin\left(\omega_i t + \Psi_i(t)\right) dt$$

In the logarithm of FLR (21) we change the unknown phases by their maximum likelihood estimates.

This procedure is reduced to the maximization of the logarithm of FLR (21) on unknown phases:

$$\begin{aligned}
L_1(\nu) = \sup_{\varphi_\nu} L_1(\nu, \varphi_\nu) = \\
= \frac{2}{N_0} \sum_{i=1}^{\nu} \left[a_i \sqrt{X_i^2 + Y_i^2} - \frac{1}{2} a_i^2 E_i \right]
\end{aligned} \tag{22}$$

Using the statistics (22), we can write down the maximum likelihood algorithm for the estimation of the number of signals (1) as:

$$\hat{\nu} = arg \sup_{\nu} L_1(\nu) \tag{23}$$

Let us consider the properties of the statistics (22) and substitute the realization of the observable data (3) in Eq. (22). Then we have the following Eq. (24):

$$L_1(\nu)=\begin{cases}\sum_{i=1}^{\nu}\left[z_i\sqrt{\begin{bmatrix}(z_i\cos\varphi_{0i}+\xi_{ci})^2+\\+(z_i\sin\varphi_{0i}+\xi_{si})^2\end{bmatrix}}-z_i^2/2\right] & \nu\le\nu_0\\ \sum_{i=1}^{\nu_0}\left[z_i\sqrt{\begin{bmatrix}(z_i\cos\varphi_{0i}+\xi_{ci})^2+\\+(z_i\sin\varphi_{0i}+\xi_{si})^2\end{bmatrix}}-z_i^2/2\right]+\\+\sum_{i=\nu_0+1}^{\nu}\left(z_i\sqrt{\xi_{ci}^2+\xi_{si}^2}-z_i^2/2\right), & \nu>\nu_0\end{cases}$$

Here:

$$\xi_{ci}=\sqrt{\frac{2}{N_0E_i}}\int_0^T n(t)\,f_i(t)\cos(\omega_i t+\Psi_i(t))\,dt$$

$$\xi_{si}=\sqrt{\frac{2}{N_0E_i}}\int_0^T n(t)\,f_i(t)\sin(\omega_i t+\Psi_i(t))\,dt$$

are mutually independent Gaussian random variables with parameters (0,1). Using the formula (24), we can calculate the abridged error probability (13) for the algorithm (23):

$$p_{\varphi a}=1-p\begin{bmatrix}z_{\nu_0}\sqrt{\begin{matrix}(z_{\nu_0}\cos\varphi_{0\nu_0}+\xi_{c\nu_0})^2+\\+(z_{\nu_0}\sin\varphi_{0\nu_0}+\xi_{s\nu_0})^2\end{matrix}}+\\-z_{\nu_0}^2/2>0\ ,z_{\nu_0+1}\sqrt{\xi_{c\nu_0+1}^2+\xi_{s\nu_0+1}^2}+\\-z_{\nu_0+1}^2/2<0\end{bmatrix}$$

After simple transformations and taking into account the independence of random variables $\xi_{c\nu_0}$, $\xi_{s\nu_0}$, $\xi_{c\nu_0+1}$, $\xi_{s\nu_0+1}$, this formula can be rearranged into the form:

$$p_{\varphi a}=1-p\Big[(z_{\nu_0}\cos\varphi_{0\nu_0}+\xi_{c\nu_0})^2+ +(z_{\nu_0}\sin\varphi_{0\nu_0}+\xi_{s\nu_0})^2>(z_{\nu_0}/2)^2\Big]\times \times p\Big[\xi_{c\nu_0+1}^2+\xi_{s\nu_0+1}^2<(z_{\nu_0+1}/2)^2\Big] \quad (25)$$

The random variable:

$$(z_{\nu_0}\cos\varphi_{0\nu_0}+\xi_{c\nu_0})^2+(z_{\nu_0}\sin\varphi_{0\nu_0}+\xi_{s\nu_0})^2$$

has non-central distribution of χ^2 with the two degrees of freedom and with non-centrality parameter $(z_{\nu_0}\cos\varphi_{0\nu_0})^2+(z_{\nu_0}\sin\varphi_{0\nu_0})^2=z_i^2$.

Therefore, the abridged error probability $p_{\varphi a}$ (25) does not depend on the value $\varphi_{0\nu_0}$. Assuming that in Eq. (25) $\varphi_{0\nu_0}=\pi/4$ is for definiteness, we obtain Eq. (26):

$$\begin{aligned}p_{\varphi a}&=1-p\left[\left(\frac{z_{\nu_0}}{\sqrt2}+\xi_{c\nu_0}\right)^2+\left(\frac{z_{\nu_0}}{\sqrt2}+\xi_{s\nu_0}\right)^2>\left(\frac{z_{\nu_0}}{2}\right)^2\right]\times\\&\times p\left[\xi_{c\nu_0+1}^2+\xi_{s\nu_0+1}^2<\left(\frac{z_{\nu_0+1}}{2}\right)^2\right]=\\&=1-F_0\left[\left(\frac{z_{\nu_0+1}}{2}\right)^2\right]+F_0\left[\left(\frac{z_{\nu_0+1}}{2}\right)^2\right]F\left[\left(\frac{z_{\nu_0}}{2}\right)^2,z_{\nu_0}\right]\end{aligned}$$

here:

$$F(x,\lambda)=\begin{cases}\frac12\int_0^x \exp\left[-(y+\lambda^2)/2\right]I_0\left(\lambda\sqrt{y}\right)dy, & x\ge0\\0, & x<0\end{cases}$$

is the distribution function of the non-central χ^2 with the two degrees of freedom and with non-centrality parameter λ^2 [12], $I_0(\cdot)$ is the modified zero-order Bessel function of the first kind, while:

$$F_0(x)=F(x,0)$$

is the distribution function of the central χ^2 with two degrees of freedom [12].

IV. The Analysis of the Estimates of the Number of Harmonic Radio Signals

Let us consider the special case when all functions from the set $\{f_i(t)\}_{i=1}^{\nu_{max}}$ are identically equal to the unit within the interval $[0,T]$, and phase modulation is absent. Then radio signals in Eq. (1) represent the harmonic oscillations segments. We are also to presuppose that the equality $\omega_k=k\cdot\omega$ holds for any $k\in\overline{1,\nu_{max}}$, where ω is a real number.

If signals in Eq. (1) satisfy the bandlimitedness condition, then the expression (8) for correlation coefficient between *i*-th and *j*-th signals will write down as:

$$\rho_{ij}=\cos(\varphi_i-\varphi_j)\sin\left[2\pi(i-j)B\right]/2\pi(i-j)B+ +\sin(\varphi_i-\varphi_j)\sin^2\left[\pi(i-j)B\right]/\pi(i-j)B \quad (27)$$

where $B = \omega T / 2\pi$.

As follows from Eq. (27), the value B determines the correlation coefficient between the signals. For example, if $i - j = 1$, $\varphi_{0i} - \varphi_{0j} = 0$ and $B = 0.3$, then $\rho_{12} = 0.5$, and for sufficiently great values of B, for example, $B > 15$, the signals in Eq. (1) are orthogonal practically.

Besides, we assume that for any i: $z_i = z$, $\Delta_i = \Delta$ and for any i, j: $\varphi_{0i} - \varphi_{0j} = 0$.

Then we should substitute the following expressions in Eq. (17):

$$R = z\left(\frac{1}{2} + \sum_{i=1}^{\nu_0 - 1} \rho_{ci\nu_0}\right) - z\sum_{i=1}^{\nu_0}\left(\rho_{ci\nu_0} \cos\Delta - \rho_{si\nu_0} \sin\Delta\right)$$
$$Q = z\left(\frac{1}{2} + \sum_{i=1}^{\nu_0} \rho_{ci(\nu_0+1)}\right) - z\sum_{i=1}^{\nu_0}\left(\rho_{ci(\nu_0+1)} \cos\Delta + -\rho_{si(\nu_0+1)} \sin\Delta\right) \quad (28)$$

instead of Eq. (16).

For the further evaluation of the formulas (17) and (28), we presuppose that $\nu_0 = 2$, $\nu_{max} = 3$, so formulas (17), (28) get the appearance:

$$p_a(z,B,\Delta) = 1 - \frac{1}{\sqrt{2\pi}} \int_0^Q exp\left(-\frac{y^2}{2}\right) \Phi\left(\frac{R + y\rho_{23}}{\sqrt{1-\rho_{23}^2}}\right) dy \quad (29)$$

$$R = z\left(\frac{1}{2} + \rho_{c12}\right) - z\sum_{i=1}^{2}\left(\rho_{ci2} \cos\Delta - \rho_{si2} \sin\Delta\right)$$
$$Q = z\left(\frac{1}{2} + \sum_{i=1}^{\nu_0} \rho_{ci3}\right) - z\sum_{i=1}^{2}\left(\rho_{ci3} \cos\Delta - \rho_{si3} \sin\Delta\right) \quad (30)$$

We choose the value $g_\varphi(\Delta)$ to describe the deterioration of quality in estimating the number of signals, due to the deviation of the expected values of phases from the true ones.

This value is determined as the relation of the abridged error probability in the case of the deviation of expected values of signal phases from their true values $p_a(z,B,\Delta)$ (29) to the abridged error probability in presence of the exactly known signal phases p_{a0} (18):

$$g_\varphi(\Delta) = p_a(z,B,\Delta) / p_{a0}$$

In Fig. 1 the dependences of the function $g_\varphi(\Delta)$ upon parameter Δ are presented under SNR $z = 4$ and with various values of the parameter B: curve 1 corresponds to $B = 0.3$, 2 – $B = 2.3$, 3 – $B = 16$.

Comparison of curves 1-3 shows the relative deterioration of the performance for the quasi-likelihood algorithm estimating the number of signals, in comparison with the maximum likelihood algorithm estimating the number of radio signals with a priori known amplitudes and phases with decreasing B, i.e. with the signal correlation score in Eq. (1) increasing.

In Fig. 2 the dependences of the function $g_\varphi(\Delta)$ in case of orthogonal signals ($B = 16$) are drawn for various SNR values z. The curve 1 corresponds to $z = 3$, 2 – $z = 4$, 3 – $z = 5$, 4 – $z = 6$, 5 – $z = 7$, 6 – $z = 8$.

Comparison of curves 1-6 shows the relative deterioration of the performance for the quasi-likelihood algorithm estimating the number of signals, in comparison with the maximum likelihood estimation algorithm with SNR increasing.

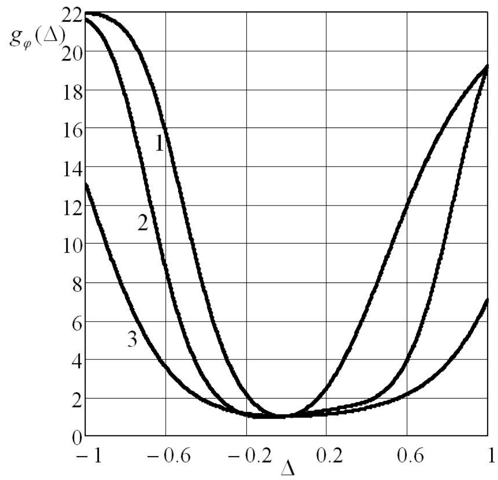

Fig. 1. The minimum gain for comparison of various feed arrangements

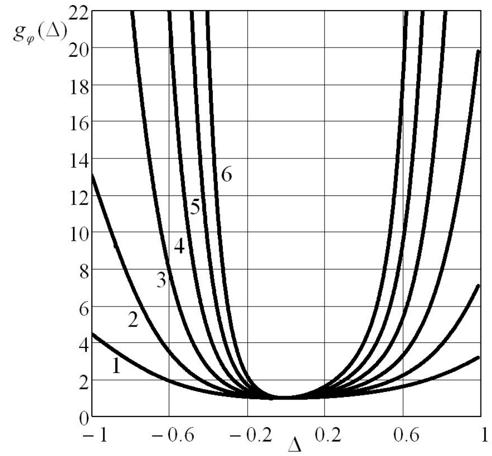

Fig. 2. The minimum gain for comparison of various feed arrangements

Further, we carry out the comparison of the quasi-likelihood (10) and maximum likelihood (23) algorithms for the estimation of the number of signals. For this purpose we introduce the value:

$$q_\varphi(\Delta) = p_{\varphi a} / p_a(z,\Delta)$$

Here $p_{\varphi a}$ is defined from Eq. (26), and $p_a(z,\Delta)$ – from Eq. (29) under $B >> 1$, i.e. if $\rho_{23} = 0$ (the condition is fulfilled when $B > 15$, it is quite enough in practice). The value q_φ shows, how many times the error probability for the quasi-likelihood algorithm (10) is less than the corresponding error probability for the maximum likelihood algorithm (24).

Dependences q_φ from value Δ are represented in Fig. 3 for various SNR values: curve 1 corresponds to SNR $z = 3$, $2 - z = 4$, $3 - z = 5$, $4 - z = 6$, $5 - z = 8$.

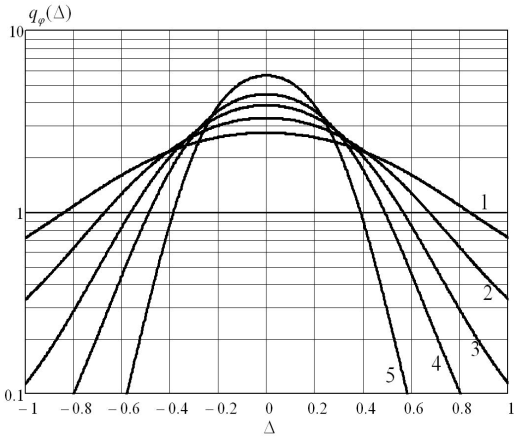

Fig. 3. The minimum gain for comparison of various feed arrangements

From the analysis of these curves follows that under $|\Delta| < 0.4$ the error probability for the quasi-likelihood estimation algorithm will be 4-6 times less (depending on SNR), than the one for the maximum likelihood estimation algorithm.

Therefore, if initial phases of radio signals are a priori known with the error of no more than 20°, then the application of the quasi-likelihood algorithm increases the efficiency of the estimation of the number of radio signals, in comparison with the efficiency of this procedure demonstrated by the maximum likelihood algorithm.

V. Conclusion

The introduced abridged error probability for the estimation of the number of signals allows us to characterize quantitatively the efficiency of the various algorithms generated for this task. The obtained results make it possible to decide between quasi-likelihood and maximum likelihood estimates of the number of signals with unknown phases.

We have shown that, for the sufficiently small lengths of the prior intervals of the allowed values of the unknown initial signal phases, the quasi-likelihood estimate of the number of signals can have the better characteristics in comparison with the corresponding maximum likelihood estimate.

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References

[1] L.I. Volkov, M.S. Nemirovsky, Yu.S. Shinakov, *Digital Radio Communication Systems [in Russian]* (Eko-Trendz, Moscow, 2005).

[2] A.G. Flaksman, Adaptive Signal Processing in Antenna Arrays with Allowance for the Rank of the Impulse-Response Matrix of a Multipath Channel, *Radiophysics and Quantum Electronics, Volume 45*, (Issue 12), *December 2002*, *Pages 977–988*.

[3] V.B. Maneilis, Algorithms for Tracking and Demodulation of the Mobile Communication Signal in the Conditions of the Indiscriminating Multipath Propagation [in Russian], *Radiotechnika*, (Issue 4), *April 2007*, *Pages 16–21*.

[4] D. Kundu, Estimating the number of signals in the presence of white noise, *Journal of Statistical Planning and Inference*, *Volume 90*, (Issue 1), *January 2000*, *Pages 57–68*.

[5] M. Wax, T. Kailath, Detection of signals by information theoretic criteria, *IEEE Trans. Acoust. Speech Signal Process*, *Volume 33*, (Issue 2), *Feb 1985*, *Pages 387–392*.

[6] A.A. Loginov, O.A. Morozov, M.Yu. Semenova, V.R. Fidelman, Method for Estimating the Number of Radiation Sources in the Problem of the Amplitude Monopulse Direction Finding, *Radiophysics and Quantum Electronics*, *Volume 56*, (Issue 7), *December 2013*, *Pages 456–463*.

[7] A.P. Trifonov, A.V. Kharin, Estimating the Number of Signals with Unknown Amplitudes [in Russian], *Nelineinyi Mir*, *Volume 11*, (Issue 12), *December 2013*, *Pages 853–866*.

[8] A.P. Trifonov, A.V. Kharin, Estimation of the Number of Radio Signals with Unknown Amplitudes and Phases, *Radiophysics and Quantum Electronics*, *Volume 58*, (Issue 1), *June 2015*, *Pages 57–70*.

[9] A.P. Trifonov, A.V. Kharin, Estimation of the number of orthogonal signals with unknown non-energy parameters, *Radioelectronics and Communications Systems*, *Volume 58*, (Issue 8), *August 2015*, *Pages 362–370*.

[10] V.I. Mudrov, V.L. Kushko, *Measurement Processing Methods. Quasi-likelihood Estimates [in Russian]* (Radio i Svyaz', Moscow, 1983).

[11] E.I. Kulikov, A.P. Trifonov, *Estimation of Signal Parameters against Hindrances [in Russian]* (Sovetskoe Radio, Moscow, 1978).

[12] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, (National Bureau of Standards. Applied Mathematics Series 55, 1964).

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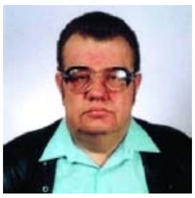

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